2. VECTORS - ALGEBRA AND GEOMETRY

September 16, 2009

1 Algebraic Definitions

For each positive integer n, an *n-vector* is a sequence $(a_1, a_2, ..., a_n)$ of n real numbers, otherwise known as an *n-tuple* of real numbers. Thus, $(2, -\sqrt{17})$ is a 2-vector and $(-1, \pi, \sin 2)$ is a 3-vector. The set of all n-vectors is called *Euclidean n-space* and is denoted by $\Re^n$. In this text we shall be concerned almost exclusively with 3-vectors. Consequently, we shall formulate all the subsequent discussion in terms of 3-vectors and refer to them as simply *vectors*.

It is customary to denote vectors by bold letters. The *components* of the vector $\mathbf{a} = (a_1, a_2, a_3)$ are the numbers a_1, a_2, a_3. The following labels are commonly used:

$$\mathbf{0} = (0, 0, 0)$$

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

Vectors are subject to the binary operations of addition and subtraction which are defined as follows:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

and

$$(a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3).$$

Thus,

$$(-1, b, \sqrt{3}) + (2, 1, \sqrt{3}) = (1, b + 1, 2\sqrt{3})$$

and

$$(-1, b, \sqrt{3}) - (2, 1, \sqrt{3}) = (-3, b - 1, 0).$$

Note that if

$$\mathbf{a} = (a_1, a_2, a_3) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, b_3) \tag{1}$$

then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (b_1 + a_1, b_2 + a_2, b_3 + a_3) \\ &= (b_1, b_2, b_3) + (a_1, a_2, a_3) = \mathbf{b} + \mathbf{a}. \end{aligned}$$

Hence the addition of vectors is commutative. It is also associative, in the sense that for any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}. \tag{2}$$

Many of the identities that hold for numbers also hold for vectors and their proof are often as straightforward as those of the commutative and associative rules given above. They will therefore be stated without fanfare and their proofs will be relegated to the exercises.

In the context of this text, *scalars* are real numbers and are denoted by roman letters $a, b, c, \ldots, r, s, t, \ldots$. These scalars have their own well known arithmetic operations which need not be belabored here. Rather, we turn to their interaction with vectors by means of the definition

$$r(a_1, a_2, a_3) = (a_1, a_2, a_3)r = (ra_1, ra_2, ra_3).$$

Thus,

$$(-2)(-1, b, \sqrt{3}) = (2, -2b, -2\sqrt{3}).$$

Note that if Eq'n (1) holds then for any vectors $\mathbf{a}, \mathbf{b}$ and scalar r

$$\begin{aligned} r(\mathbf{a} + \mathbf{b}) &= r(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (r(a_1 + b_1), r(a_2 + b_2), r(a_3 + b_3)) \\ &= (ra_1 + rb_1, ra_2 + rb_2, ra_3 + rb_3) \\ &= r(a_1, a_2, a_3) + r(b_1, b_2, b_3) = r\mathbf{a} + r\mathbf{b} \end{aligned}$$

and for any scalars s, t

$$(s + t)\mathbf{a} = s\mathbf{a} + t\mathbf{a} \tag{3}$$

$$(rs)\mathbf{a} = r(s(\mathbf{a})). \tag{4}$$

In other words, scalar multiplication of vectors is distributive and associative. It is clear that

$$(a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

EXERCISES 2.1

1. Prove Eq'ns (2), (3), (4).

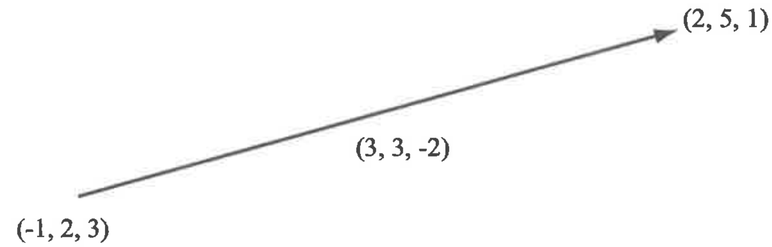

Figure 1: A geometric vector

2 Geometric Interpretation

Vectors can be endowed with a very useful geometric interpretation. We begin with an arbitrary Cartesian coordinate system. The vector (a_1, a_2, a_3) denotes any of the directed line segments PQ where P has coordinates (p_1, p_2, p_3), Q has coordinates (q_1, q_2, q_3) and

$$q_1 - p_1 = a_1, q_2 - p_2 = a_2, q_3 - p_3 = a_3 \tag{5}$$

(Figure 1). We shall refer to these geometric representations as *geometric vectors* as opposed to the strictly numerical or *algebraic* vectors introduced above. The geometric vector from P to Q is denoted by $\vec{PQ}$. Thus, each fixed algebraic vector is represented by an infinitude of geometric vectors - one emanating from each point in space. In particular, the algebraic vector (p_1, p_2, p_3) is represented by the geometric vector $\vec{OP}$ from the origin O to the point P. Two geometric vectors are said to be *equal* if they represent the same algebraic vector. This is tantamount to saying that if $P = (p_1, p_2, p_3), Q = (q_1, q_2, q_3), R = (r_1, r_2, r_3), S = (s_1, s_2, s_3)$ (see Figure 2) then the geometric vectors $\vec{PQ}$ and $\vec{RS}$ are equal if and only if

$$(q_1 - p_1, q_2 - p_2, q_3 - p_3) = (s_1 - r_1, s_2 - r_2, s_3 - r_3). \tag{6}$$

Example 2.1

If $P = (2, 3, 4), Q = (0, 5, -1), R = (-1, 2, 0), S = (-3, 4, -5)$ then

$$\vec{PQ} = \vec{RS} = (-2, 2, -5) \quad \text{and} \quad \vec{PR} = \vec{QS} = (-3, -1, -4).$$

Proposition 2.2 *Two geometric vectors $\vec{PQ}$ and $\vec{RS}$ are equal if and only if they satisfy the following constraints:*

i. they have equal lengths;

ii. the infinite straight lines that underlie them are either parallel or identical;

iii. they have the same directions.

PROOF: Suppose the two geometric vectors PQ and RS (Figure 3) are equal. By Eq'n (6) above and the distance formula of Chapter 1, it follows

3 The Dot Product

Vector multiplication can be defined in (at least) two ways both of which have some analogs of scalar multiplication and both of which display some "odd" behavior. The first of these is the *dot* product

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

For example

$$(-1, 2, -3) \cdot (2, 5, -1) = -2 + 10 + 3 = 11$$

and

$$(-1, 2, 2) \cdot (10, 2, 3) = 0.$$

It is easy to see that this product is both commutative i.e.,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \tag{7}$$

and distributive i.e,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \tag{8}$$

and

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}. \tag{9}$$

However, since $\mathbf{a} \cdot \mathbf{b}$ is a scalar and not a vector, the iterated product $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ is undefined. Hence the associative law does not hold for the dot product (see Exercise 1). In fact, the closure law is also not satisfied since the dot product of two vectors is not a vector. The proof of the following equations is relegated to the exercises.

$$r(\mathbf{a} \cdot \mathbf{b}) = (r\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (r\mathbf{b}). \tag{10}$$

Proposition 3.1 *If* $\mathbf{a} = (a_1, a_2, a_3)$ *is an algebraic vector, then the length of its geometric representation is*

$$\sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

PROOF: Any geometric representative $\vec{PQ}$ of $\mathbf{a}$ has endpoints $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ such that

$$q_i - p_i = a_i, \ i = 1, 2, 3.$$

Hence the length of PQ is

$$\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$$

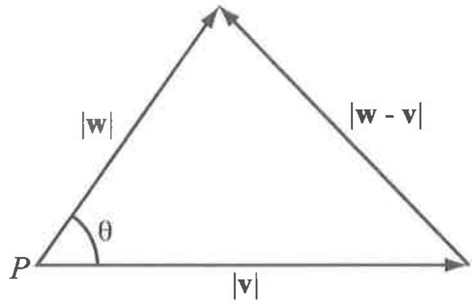

Figure 5: The dot product

$$= \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

Q.E.D.

The *length* of the algebraic vector **a**, denoted by $|\mathbf{a}|$, is defined to be the length of any of its geometric representatives. I. e.,

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

The *angle* between two vectors is defined to be the angle between any of their intersecting representatives. We shall soon see that this angle is indeed well defined.

Proposition 3.2 *Let θ be the angle between the geometric representatives of* **v** *and* **w** *that are both anchored at some point P. Then*

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \theta.$$

PROOF: Apply the Law of Cosines to Figure 5 to get

$$|\mathbf{w} - \mathbf{v}|^2 = |\mathbf{w}|^2 + |\mathbf{v}|^2 - 2|\mathbf{w}||\mathbf{v}| \cos \theta$$

or

$$(\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) = \mathbf{w} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{w}||\mathbf{v}| \cos \theta$$

or. By the distributive law of the dot product,

$$\mathbf{w} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{w}||\mathbf{v}| \cos \theta$$

and hence, by the commutativity of the dot product,

$$-2\mathbf{v} \cdot \mathbf{w} = -2|\mathbf{w}||\mathbf{v}| \cos \theta$$

from which the proposition follows immediately. Q.E.D.

For example, the angle between the vectors (-1, 2, -3) and (2, 1, -1) is

$$\cos^{-1}\frac{-2+2+3}{\sqrt{14}\sqrt{6}} = \cos^{-1}\frac{3}{\sqrt{84}}.$$

Corollary 3.3 *The angle between two algebraic vectors is well defined.*

PROOF: This follows from the observation that the point P did not play any role whatsoever in the proof of Proposition 3.2. Q.E.D.

Two vectors are said to be *orthogonal* if the angle between them is a right angle.

Corollary 3.4 *Two vectors are orthogonal if and only if their dot product is zero.*

PROOF: This follows immediately from Proposition 3.2. Q.E.D.

A vector $\mathbf{v}$ such that

$$|\mathbf{v}| = 1$$

is a *unit vector.* The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as well as

$$(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}) \quad \text{and} \quad (\cos\theta, 0, \sin\theta)$$

are all unit vectors.

Proposition 3.5 *If* $\mathbf{v}$ *is any vector, then*

$$\frac{\mathbf{v}}{|\mathbf{v}|}$$

is a unit vector.

PROOF: See Exercise 4. □

The projection of a vector $\mathbf{v}$ onto the vector $\mathbf{u}$, denoted by $\text{proj}_{\mathbf{u}}\mathbf{v}$ is defined by Figure 6 where

$$\mathbf{u} = \vec{PQ}, \quad \text{and} \quad RH \perp PQ.$$

It is easily seen (Exercise 5) that

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}. \tag{11}$$

For example,

$$\text{proj}_{\mathbf{i}}\mathbf{v} = \frac{\mathbf{i}\cdot\mathbf{v}}{\mathbf{i}\cdot\mathbf{i}}\mathbf{i} = \frac{v_1}{1}\mathbf{i} = (v_1, 0, 0).$$

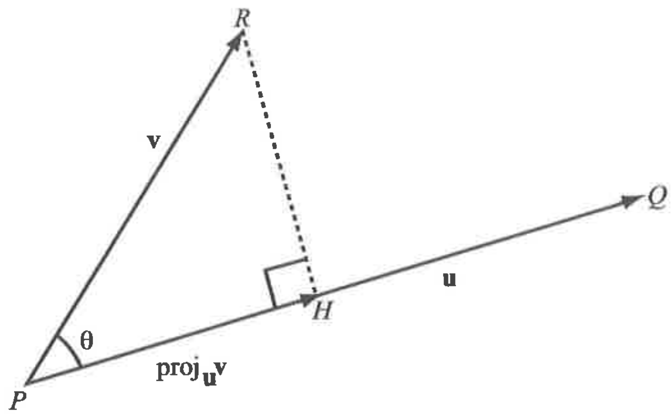

Figure 6: The projection of one vector onto another

EXERCISES 2.3

1. Prove the Cauchy-Schwartz inequality:

$$|\mathbf{u}\cdot\mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|.$$

2. Prove the Triangle Inequality:

$$|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|.$$

3. Prove the Parallelogram Law:

$$|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2.$$

4. Find the projection of (1,2,3) onto (3, -2, 1).
5. Prove that

$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}.$$

6. Prove Eq'ns (8), (9), (10).
7. Prove Eq'n (11).
8. Suppose we extend the definition of the dot product by defining

$$r\cdot\mathbf{a} = \mathbf{a}\cdot r = r\mathbf{a}.$$

Is this extended dot product associative?

4 The Cross Product

The *cross product* of the vectors **a** and **b** is denoted and defined by

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

For example

$$
\begin{gathered}
(-1,2,0) \times (2,-3,5) \\
= (2 \cdot 5 - 0 \cdot (-3), 0 \cdot 2 - (-1) \cdot 5, (-1) \cdot (-3) - 2 \cdot 2) \\
= (10, 5, -1).
\end{gathered}
$$

Some of the identities that involve this product are

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{12}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \text{ if and only if } \mathbf{a} = s\mathbf{b} \text{ or } \mathbf{b} = s\mathbf{a} \text{ for some scalar } s \tag{13}$$

$$r\mathbf{a} \times s\mathbf{b} = rs(\mathbf{a} \times \mathbf{b}) \tag{14}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \tag{15}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \tag{16}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{17}$$

$$(\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b}) = (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) - (\mathbf{a}, \mathbf{b})^2 \tag{18}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}. \tag{19}$$

We now establish some useful identities that relate the dot and cross products to each other.

Proposition 4.1 (Lagrange) *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{d}$ *are any four 3-vectors, then*

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \tag{20}$$

PROOF: By the distributivity of both the dot and the cross products

$$((\mathbf{a} + \mathbf{a}') \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) + (\mathbf{a}' \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$$

$$
\begin{gathered}
((\mathbf{a} + \mathbf{a}') \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - ((\mathbf{a} + \mathbf{a}') \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \\
= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) + (\mathbf{a}' \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a}' \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})
\end{gathered}
$$

$$
\begin{gathered}
(s(\mathbf{a}) \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = s[(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})] \\
((s\mathbf{a}) \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - ((s\mathbf{a}) \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \\
= s[(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})]
\end{gathered}
$$

and that similar equations hold for $\mathbf{b}, \mathbf{c}$ and $\mathbf{d}$. It follows that it suffices to prove the proposition when each of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ is one of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Note also that interchanging $\mathbf{a}$ with $\mathbf{b}$ merely reverses the sign of each side of Eq'n (20) as does interchanging $\mathbf{c}$ and $\mathbf{d}$.

If $\mathbf{a} = \mathbf{b}$ or $\mathbf{c} = \mathbf{d}$, then the desired Eq'n (20) reduces to $0 = 0$. Hence it may be assumed that $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{c} \neq \mathbf{d}$. If $\{\mathbf{a}, \mathbf{b}\} = \{\mathbf{c}, \mathbf{d}\}$ Eq'n (20) reduces to

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})$$

which is easily verified (keep in mind that $\{\mathbf{a}, \mathbf{b}\}$ consist of two distinct vectors of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and so there are only 3 cases to consider). If $\{\mathbf{a}, \mathbf{b}\} \neq \{\mathbf{c}, \mathbf{d}\}$ note that both of the sides of Eq'n (20) only change their signs when $\mathbf{a}$ is interchanged with $\mathbf{b}$ as well as when $\mathbf{c}$ is interchanged with $\mathbf{b}$. Hence it may be assumed that

$$\{\mathbf{a}, \mathbf{b}\} \cap \{\mathbf{c}, \mathbf{d}\} = \mathbf{a} = \mathbf{c}.$$

Consequently it suffices to verify Eq'n (20) in the following chases

a	b	c	d
i	j	i	k
i	k	i	j
j	i	j	k
j	k	j	i
k	i	k	j
k	j	k	i

which is easily done. Q.E.D.

Proposition 4.2 *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be two algebraic vectors. Then* $\mathbf{v} \times \mathbf{w}$ *has the properties:*

i. $|\mathbf{v} \times \mathbf{w}|$ *equals the area of the parallelogram spanned by the representatives of* $\mathbf{v}$ *and* $\mathbf{w}$*,*

ii. $\mathbf{v} \times \mathbf{w}$ *is orthogonal to both* $\mathbf{v}$ *and* $\mathbf{w}$*,*

PROOF: i. By the identity of Lagrange

$$(\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - (\mathbf{v} \cdot \mathbf{w})^2$$
$$= |\mathbf{v}|^2|\mathbf{w}|^2 - (|\mathbf{v}||\mathbf{w}|\cos\theta)^2 = |\mathbf{v}|^2|\mathbf{w}|^2(1 - \cos^2\theta)$$
$$= (|\mathbf{v}||\mathbf{w}|\sin\theta)^2.$$

Since $0 \leq \theta \leq \pi$ it follows from a well known trigonometric formula that the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ is

$$|\mathbf{v}||\mathbf{w}|\sin\theta = |\mathbf{v} \times \mathbf{w}|.$$

ii. By Eq'n (17)

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{v}) = \mathbf{w} \cdot \mathbf{0} = 0.$$

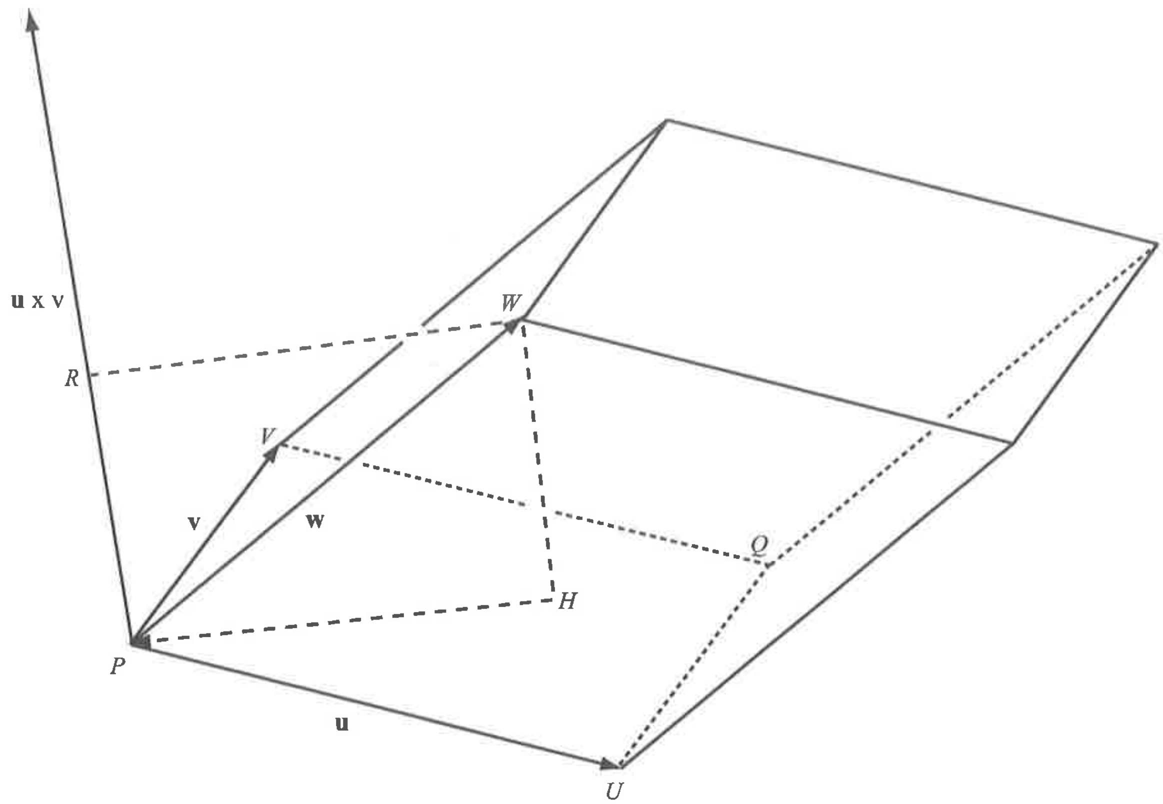

Figure 7: Volume of parallelopiped

By Corollary 3.4 $\mathbf{v}$ (and similarly $\mathbf{w}$) is orthogonal to $\mathbf{v} \times \mathbf{w}$. Q.E.D.

A *parallelopiped* is a boxlike solid in which the right angles of the box have been replaced by arbitrary angles, in such a manner that opposite faces remain parallel to each other. If the geometric vectors formed by the edges emanating from the same vertex of the parallelopiped are denoted by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, then the parallelopiped is said to be spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. The opposite faces of the parallelopiped arre parallel and each of those faces is a parallelogram. The volume of a parallelopiped equals the product of the area of a face with the distance between a face and its opposite face.

Proposition 4.3 *If* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are three algebraic vectors, then the volume of the parallelopiped spanned by their representatives at any point is*

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|.$$

PROOF: The volume of the parallelopiped of Figure 7 equals the product of the area of its base $PUQV$ and its altitude WH. By Proposition 4.2 this equals

$$|\mathbf{u} \times \mathbf{v}| \cdot WH.$$

Let WR be perpendicular to $\mathbf{u} \times \mathbf{v}$. Since both $\mathbf{u} \times \mathbf{v}$ and WH are perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$, it follows that

$$WH = PR = |\text{proj}_{\mathbf{u}\times\mathbf{v}}(\mathbf{w})| = \left| \frac{(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}}{(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v})} \mathbf{u} \times \mathbf{v} \right|$$

Thus, the volume of the parallelopiped equals

$$|\mathbf{u} \times \mathbf{v}| \frac{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}{|\mathbf{u} \times \mathbf{v}|} = |\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|.$$

Q.E.D.

Example 4.4 *The vectors (1, 2, 3), (2, 3, 1), (3, 1, 2) span a parallelopiped whose volume is*

$$|((1,2,3) \times (2,3,1)) \cdot (3,1,2)|$$
$$= |(-7,5,-1) \cdot (3,1,2)| = |-21+5-2| = 18.$$

Proposition 4.5 *The vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *form a right handed system if and only if* $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} > 0$.

PROOF: Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w}$ be anchored at the origin. We assume that $\mathbf{u}$ and $\mathbf{v}$ are in *standard position* in the sense that

$$u_3 = v_3 = 0, \quad v_2 > 0, \quad u_2 = 0, \quad u_1 > 0. \tag{21}$$

If θ is the counterclockwise angle from $\mathbf{u}$ to $\mathbf{v}$ then $0 < \theta < \pi$ so that $\sin\theta > 0$ and hence

$$\mathbf{v} = |\mathbf{v}|(\cos\theta, \sin\theta, 0).$$

Consequently,

$$\mathbf{u} \times \mathbf{v} = (u_1, 0, 0) \times |\mathbf{v}|(\cos\theta, \sin\theta, 0) = u_1|\mathbf{v}|(0, 0, \sin\theta).$$

and hence

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = w_3 u_1 |\mathbf{v}| \sin\theta.$$

Note that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in standard position form a right handed system if and only if $w_3 > 0$ which is tantamount to $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} > 0$ because

$$u_1 |\mathbf{v}| \sin\theta > 0.$$

Thus the proposition is proved when the vectors are in standard position.

Next suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in general position and rotate them until they fall respectively onto $\mathbf{u}', \mathbf{v}', \mathbf{w}'$ in standard position. It is clear that the "handedness" of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the same as that of $\mathbf{u}', \mathbf{v}', \mathbf{w}'$. By Proposition 4.3 the fraction

$$\frac{(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}}{volume(\mathbf{u}, \mathbf{v}, \mathbf{w})}$$

has the value 1 or -1 at each position of the triple $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Since this fraction varies continuously as this triple is rotated, it must constantly be 1 or constantly be -1. Consequently, the same must hold for the sign of the numerator $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

Thus, both the "handedness" of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and the sign of $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ remain invariant when these vectors are subjected to a rotation. Since the equivalence of right handedness and the positiveness of $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ hold in standard position, and since every triple of vectors can be rotated into standard position, the theorem is proved in general position. Q.E.D.

Corollary 4.6 *If neither* $\mathbf{u}$ *nor* $\mathbf{v}$ *is a scalar multiple of the other, then* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{u} \times \mathbf{v}$ *form a right handed system.*

PROOF: This follows from Proposition 4.5 upon setting $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. Q.E.D.

EXERCISES 2.4

1. Complete the proof of Proposition 5.
2. Prove Eq'ns (12), (14), (15), 16).
3. Prove Eq'n (13).
4. Prove Eq'ns (17), (18).
5. Prove Eq'n (19).
6. Prove Proposition 4.1 using a computer.
7. Find the volume of the parallelopiped spanned by the vectors (1, 2, 3), (-1, 3, 5), (3, -2, 4) at the origin.
8. Find the volume of the tetrahedron with vertices (1, 2, 3), (-1, 3, 5), (1, -3, 5), (-1, -3, 5).
9. Find the area of the parallelogram spanned by the vectors (1, 2, 3) and (-1, 3, 5).
10. Find the area of the triangle spanned by the the points (1, 2, 3), (-1, 3, 5), (3, -2, 4).
11. Let $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3), \mathbf{c} = (c_1, c_2, c_3)$. Prove that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

(See Figure 8 for a mnemonic devide.)

12. Which of the following triples of vectors are right handed and which are left handed?
   a. (1, 2, 2), (-1,0, 1), (1, 1, 1)
   b. (3, 1, 1), (-1, 0, 1), (1, 1, 1)
   c. (-2, -1, 0), (-1, 0, 1), (1, 1, 1)
   d. (1, 3, 5), (-1, 0, 1), (1, 1, 1).

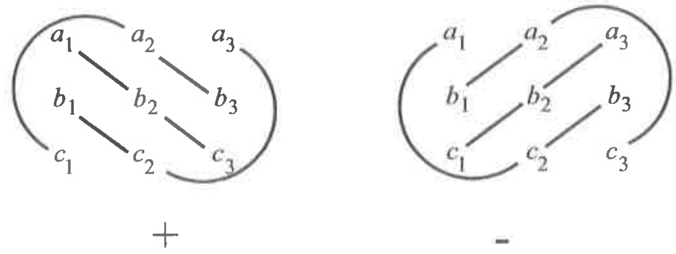

Figure 8: The determinant

5 Vector Equations of Lines and Planes

Let n be a straight line in a Cartesian coordinate system with direction angles $(\alpha_1, \alpha_2, \alpha_3)$ and a fixed but arbitrary point $A = (a_1, a_2, a_3)$ on n . We saw in Chapter 1 Proposition 3.1, that when s denotes the distance from a variable point $\mathbf{x} = (x_1, x_2, x_3)$ on n, the three coordinates of P satisfy the equations

$$x_i = a_i + s\cos\alpha_i, \quad i = 1, 2, 3$$

or, in vector terminology,

$$\mathbf{x} = \mathbf{a} + s\mathbf{d} \quad \text{where} \quad \mathbf{d} = (\cos\alpha_1, \cos\alpha_2, \cos\alpha_3). \tag{22}$$

If (m_1, m_2, m_3) are any direction numbers of n and $\mathbf{m} = (m_1, m_2, m_3)$ then, by Proposition 2.2, the direction cosines of n are

$$\cos\alpha_i = \frac{m_i}{|\mathbf{m}|} \quad i = 1, 2, 3$$

or

$$\mathbf{d} = (\cos\alpha_1, \cos\alpha_2, \cos\alpha_3) = \frac{\mathbf{m}}{|\mathbf{m}|}$$

If we set

$$t = \frac{s}{|\mathbf{m}|}$$

the Eq'n(23) is transformed into

$$\mathbf{x} = \mathbf{a} + t|\mathbf{m}|\frac{\mathbf{m}}{|\mathbf{m}|} = \mathbf{a} + t\mathbf{m}$$

or

$$\mathbf{x} = \mathbf{a} + t\mathbf{m}. \tag{23}$$

This is the *vector equation* of the straight line that contains the point $\mathbf{a}$ and whose direction numbers are the components of $\mathbf{m}$.

Example 5.1 *The y-axis contains the origin and has direction cosines (0,1, 0). Hence it has the vector equation*

$$\mathbf{x} = \mathbf{0} + t(0, 1, 0) = (0, t, 0).$$

Example 5.2 *The vector equation of the line through (1, -1, 2) with direction numbers (1, 2, -3) is*

$$\mathbf{x} = (1, -1, 2) + t(1, 2, -3) = (1 + t, -1 + 2t, 2 - 3t).$$

Example 5.3 *The vector equation of the straight line joining the points* **a** *and* **b** *is*

$$\mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = (1 - t)\mathbf{a} + t\mathbf{b}.$$

By Proposition 1.4.1, the equation whose graph is the plane normal to the straight line m with direction numbers (m_1, m_2, m_3) and containing the point (a_1, a_2, a_3) on m is

$$(x - a_1)m_1 + (y - a_2)m_2 + (z - a_3)m_3 = 0.$$

If we set $\mathbf{m} = (m_1, m_2, m_3), \mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{x} = (x, y, z)$ this becomes the vector equation

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{m} = 0.$$

Example 5.4

EXERCISES 2.5

1. Find the vector equation of the straight line that joins the points (1, 2, 3) and (7, -2, -1).
2. Find the vector equation of the plane that contains the three points (1, 2, 3), (7, -2, -1) and (2, 2, 2).

6 Vector Valued Functions

A vector function $\mathbf{F}(x) : D \subset \Re \to \Re^3$ is said to be differentiable if each of its components is a differentiable function of x and we write

$$\mathbf{F}'(x) = \frac{d\mathbf{F}}{dx} = (F_1'(x), F_2'(x), F_3'(x)).$$

This operation satisfies the rules

$$(\mathbf{F} + \mathbf{G})' = \mathbf{F}' + \mathbf{G}' \tag{24}$$

$$(k\mathbf{F})' = k\mathbf{F}' \tag{25}$$

$$(f\mathbf{F})' = f'\mathbf{F} + f\mathbf{F}' \tag{26}$$

$$(\mathbf{F} \cdot \mathbf{G})' = \mathbf{F}' \cdot \mathbf{G} + \mathbf{F} \cdot \mathbf{G}' \tag{27}$$

$$(\mathbf{F} \times \mathbf{G})' = \mathbf{F}' \times \mathbf{G} + \mathbf{F} \times \mathbf{G}' \tag{28}$$

EXERCISES 2.6

1. Prove Eqns (25) - (29).