6. TRANSFORMATIONS

1 Examples

The reader is reminded of the all-improtant method of integration by substitution. One way to state this is that if x is a one-to-one function of u, then

$$\int f(x)dx = \int f(x(u))\frac{dx}{du}du. \tag{1}$$

Example 1.1 *Evaluate*

$$\int_0^1 2x\sqrt{1+x^2}dx.$$

yyyexample to explain that one-to-one is important

The transformation of the integral in (yyy) is effected by replacing x and dx with $x(u)$ and du and introducing the new factor dx/du. We now look for a two dimensional analog of this process. In other words, we are looking for a way of transforming integrals of the form

$$\int\int f(x,y)dxdy \tag{2}$$

It stands to reason that such an analog would call for a substitution

$$x = x(u,v), \quad y = y(u,v). \tag{3}$$

Such a pair of equations is called a *transformation* and is denoted by

$$(x,y) = T(u,v).$$

We digress to discuss the geometry of some examples of such transformations. One of the ways of visualizing such transformations is by means of *isobars.* These are the curves that consist of the graphs of the equations

$$x(u,v) = c, \quad y(u,v) = d$$

in the uv-plane, where c and d are arbitrary constants.

Example 1.2 *Draw the isobars of the transformation*

$$x = u, \quad y = u + v$$

that correspond to $x, y = -2, -1, 0, 1, 2$.

The isobar that corresponds to a fixed value of $x = c$ is the straight line

$$u = c.$$

This is the vertical straight line of the uv-plane that intersects the u-axis at the point (c, 0). (See Fig. 1) The isobar that corresponds to a fixed value of $y = d$ is the straight line

$$u + v = d.$$

This is the straight line of slope -1 that intersects the u-axis at $(d, 0)$ and the v-axis at the point $(0, d)$. (See Fig. 1)

Example 1.3 *Draw the isobars of the transformation*

$$x = u^2 - v^2, \quad y = uv$$

that correspond to $x, y = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

For $c \neq 0$ the isobar that corresponds to $x = c$ is a hyperbola whose asymptotes are the straight lines $v = \pm u$ (Figure 2). For $c = 0$ the isobar is the union of the two straight lines $v = \pm u$. For $d \neq 0$ the isobar that corresponds to $y = d$ is a hyperbola whose asymptotes are the coordinate axes (Figure 2). For $d = 0$ the isobar is the union of the two coordinate axes.

2 The Jacobian

Working by analogy with the one-dimensional process this double integral of (2) would be transformed into an expression of the form

$$\int\int f(x(u,v), y(u,v))J(u,v)dudv$$

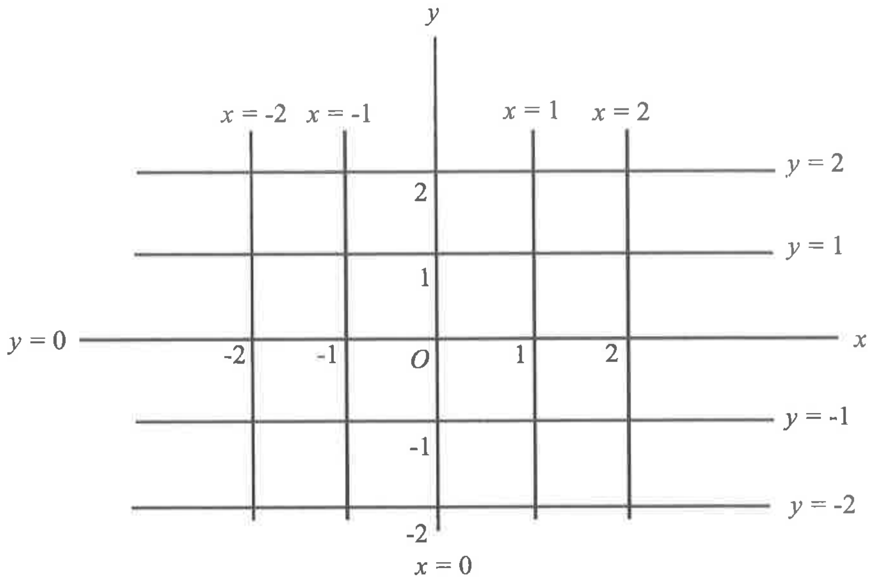
where $J(u,v)$ is some two-dimensional analog of dx/du.

Informal argument:

Let $f(u,v) : D \subset \Re^2 \to \Re$. We begin by reminding the readers that, by definition.

$$\int\int_D f dA = \lim \sum_{i=0}^{n} f(u_i, v_i)\Delta A_i$$

where $\{A_1, A_2, \ldots, A_n\}$ constitutes a partition of D into (small) regions, (u_i, v_i) is an arbitrary point in D_i, ΔA_i denotes the area of A_i and the limit



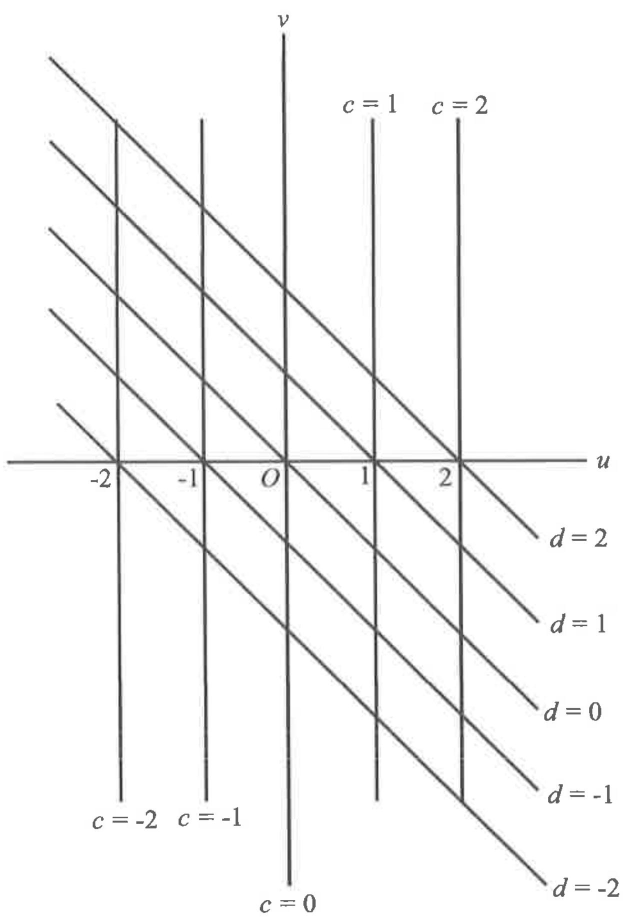


Figure 1: The transformation $x = u, y = u + v$

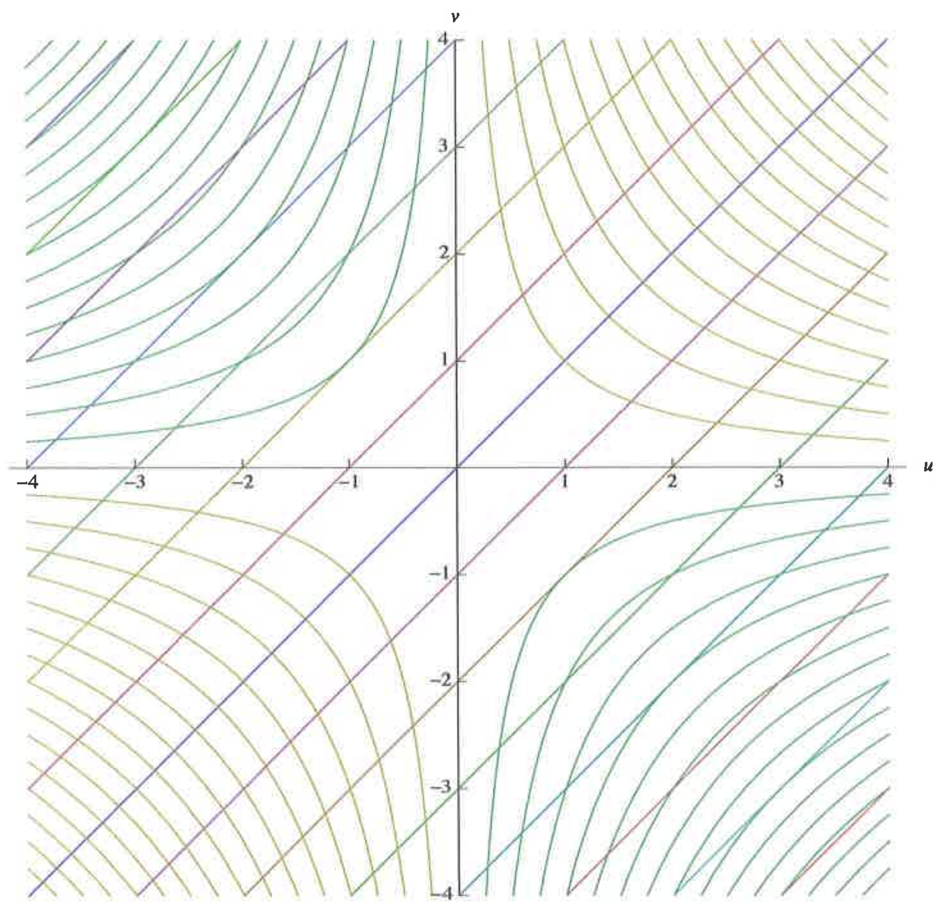


Figure 2: The transformation $x = u^2 - v^2, y = uv$

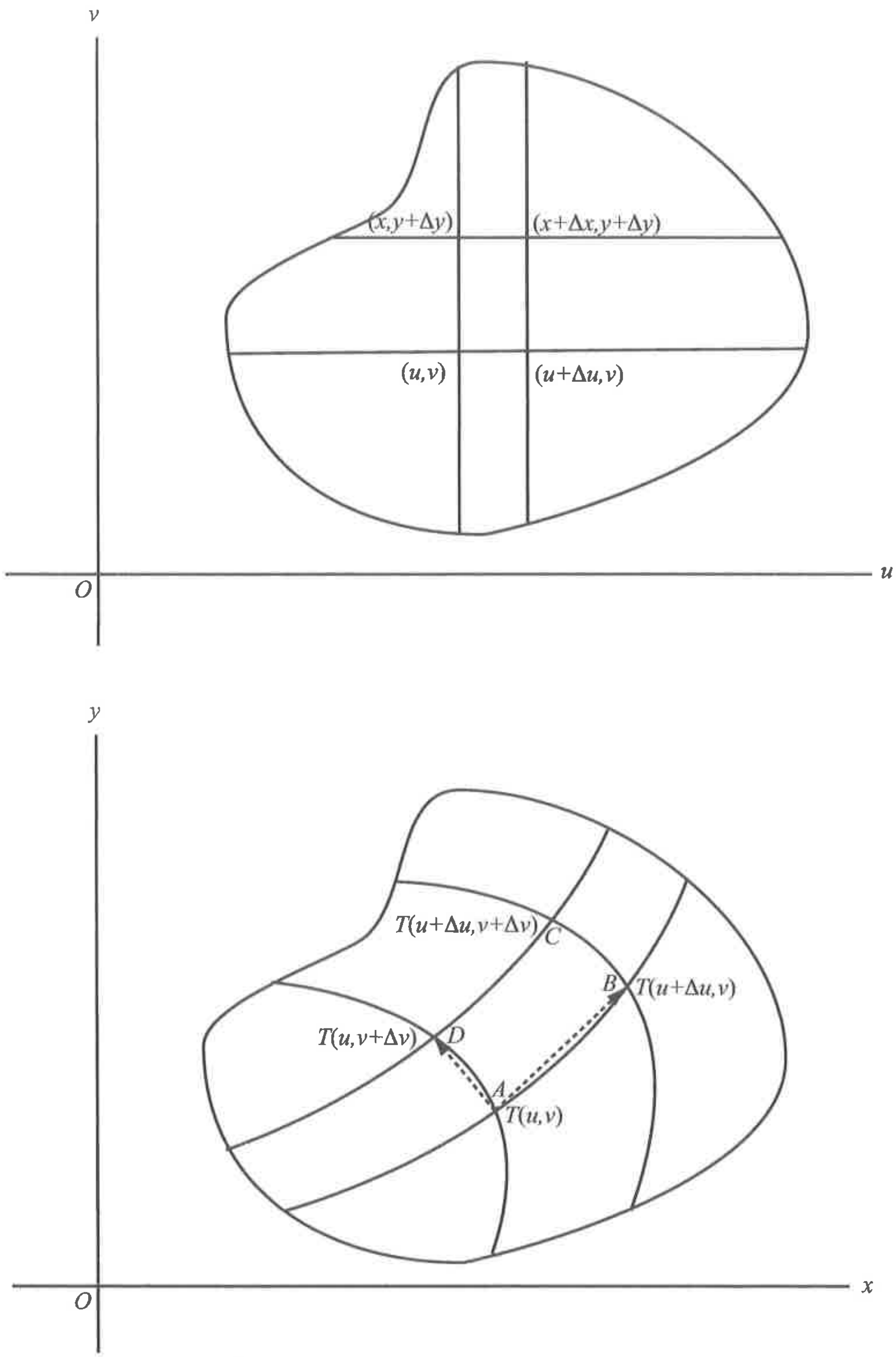


Figure 3: The transformation of an area element

is taken as the diameters of the $A_i's$ converge to 0. In the case where the partition consists of rectangles formed by vertical and horizontal lines this sum equals

$$\int\int_D f(u,v)dudv.$$

It may happen that the integrand $f(u,v)$ can be factored as $g \circ T$. For example, if T is the transformation of Example 2, then

$$f(u,v) = uv\sqrt{u^2 - v^2} = y\sqrt{x}$$

where $g(x,y) = y\sqrt{x}$.

In that case,

$$\int\int_D f(u,v)dA = \int\int_D g \circ T(u,v)dA$$

However, when the partition is formed by the isobars of T, the transformation of (1), then the area of the typical element $PQRS$ (see Fig. xxx) is approximated by the area of the parallelogram formed by $\vec{PQ}$ and $\vec{PS}$. By the Mean Value Theorem

$$\begin{aligned}\vec{PQ} &= (x(u+\Delta u, v) - x(u,v), y(u+\Delta u, v)) - y(u,v)) \\ &\approx \left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u\right) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)\Delta u\end{aligned}$$

$$\begin{aligned}\vec{PS} &= (x(u, v+\Delta v) - x(u,v), y(u, v+\Delta v)) - y(u,v)) \\ &\approx \left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v\right) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right)\Delta v.\end{aligned}$$

By yyy the area ΔA of this parallelogram is

$$\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Delta u \Delta v.$$

Consequently

$$\begin{aligned}\int\int_D f(u,v)dudv &= \int\int_D f dA \\ &= \int\int_D g \circ T(u,v)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)dudv.\end{aligned}$$

It is customary to employ the abbreviation

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

so that the integral transformation becomes

$$\int\int_D f(u,v)dudv = \int\int_D g \circ T(u,v)\frac{\partial(x,y)}{\partial(u,v)}dudv.$$

yyyexamples

The case of 3 variables is entirely analogous to the above. We stipulate a function $f(x, y, z) : D \subset \Re^3 \to \Re$ and a transformation of variables

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w).$$

The corresponding volume element is approximated by the volume of the parallelopiped spanned by the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ where

$$A = (x(u, v, w), y(u, v, w), z(u, v, w))$$

$$B = (x(u + \Delta u, v, w), y(u + \Delta u, v, w), z(u + \Delta u, v, w))$$
$$C = (x(u, v + \Delta v, w), y(u, v + \Delta v, w), z(u, v + \Delta v, w))$$
$$D = (x(u, v, w + \Delta w), y(u, v, w + \Delta w), z(u, v, w + \Delta w)).$$

By the Mean Value Theorem

$$\vec{AB} = (x(u + \Delta u, v, w) - x(u, v, w),$$

$$y(u + \Delta u, v, w) - y(u, v, w), z(u + \Delta u, v, w) - z(u, v, w))$$
$$\approx \left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, \frac{\partial z}{\partial u}\Delta u\right) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\Delta u$$

and similarly,

$$\vec{AC} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\Delta v$$

$$\vec{AD} = \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right)\Delta w$$

The volume of the 3-dimensional element is therefore

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}\Delta u \Delta v \Delta w$$

where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \cdot \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \times \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right)$$

The triple integral is thus transformed by the rule

$$\int\int\int_R f(x, y, z)dxdydz =$$

$$\int\int\int_{R'} f(x(u, v, w), y(u, v, w), z(u, v, w))\frac{\partial(x, y, z)}{\partial(u, v, w)}dudvdw$$