7. INFINITE SERIES
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1 A Historical Introduction

The oldest infinite series whose summation has been recorded is

$$1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots = \frac{4}{3}$$

which Archimedes used to evaluate the area of the segment cut from a parabola by a straight line. The care that he took to prove the validity of this infinite summation indicates that this idea was his own invention, for, in the same book, when using known results, he was content to cite his predecessors' work and did not take the trouble to reprove them. His method generalizes easily to a proof of the well known summation of the *geometric progression*

$$1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r} \quad \text{provided} \quad -1 < r < 1. \qquad (1)$$

No other infinite series is known to have been summed until the fourteenth century when Nicole Oresme (ca. 1323 - 1382) summed the series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots.$$

The development of calculus in the seventeenth century renewed mathematician's interest in such series and Newton, Leibniz, Gregory and others derived many such summations. Their methods almost always made use of questionable analogies to operations on polynomials. For example, if the term r of (1) is replaced with $-x^2$ we obtain

$$1 - x^2 + x^4 - x^6 + - \ldots = \frac{1}{1+x^2} \quad \text{provided} \quad -1 < x < 1.$$

Integration then yields

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \tan^{-1} x + C, \quad -1 < x < 1.$$

The substitution $x = 0$ tells us that $C = 0$ and hence

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \tan^{-1} x, \quad -1 < x < 1. \tag{2}$$

The restriction $-1 < x < 1$ notwithstanding, it is tempting to substitute $x = 1$ into this equation and obtain

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \tan^{-1} 1 = \frac{\pi}{4}.$$

This equation happens to be valid although the above argument cannot be considered to be a rigorous proof. Consider for example what would happen if we substituted $x = -1$ into Eq'n (1). This would result in the "equation"

$$1 - 1 + 1 - 1 + \ldots = \frac{1}{2}$$

which is clearly absurd. Still, it would be nice to have some such expression as (2) since the definition of the inverse tangent function is not much help if we wish to compute a such as $\tan^{-1} .5$ (Exercises 3, 4).

To take another example, in order to obtain the infinite series expansion of the function $y = e^x$ we might start out by assuming such a series to exist, say

$$y = e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots. \tag{3}$$

Differentiation yields

$$y' = e^x = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$$

The term by term comparison of these two series yields

$$c_n = \frac{c_{n-1}}{n} \quad = 1, 2, 3, \ldots.$$

Since the substitution $x = 0$ into Eq'n (3) gives

$$c_0 = e^0 = 1$$

it follows that

$$c_n = \frac{1}{n!}. \tag{4}$$

Thus,

$$e^x = \sum_{i=0}^{\infty} \frac{x^n}{n!}. \tag{5}$$

Many other examples will be found in the exercises.

In the early eighteenth century Brook Taylor observed that if a function $f(x)$ does have a power series expansion, say,

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \tag{6}$$

then the coefficient c_n can be evaluated by the differentiating this equation n times and substituting $x = a$. This yields

$$f^{(n)}(a) = n!c_n$$

or

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

For example, if $f(x) = e^x$ and $a = 0$, then

$$c_n = \frac{e^{(n)}(0)}{n!} = \frac{1}{n!} \tag{7}$$

which agrees with Eq'n (4). The substitution of Eq'n (7) into Eq'n (6) yields the *Taylor series* or *Taylor expansion*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{8}$$

The particular case of Eq'n (7) obtained by setting $a = 0$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{9}$$

is called the *Maclaurin series*.

While these formulas looks very powerful, they do suffer from some disadvantages. To begin with, they are sometimes incorrect (see Exercise 8). In addition, the computation of all the derivatives of a function is, generally speaking, a difficult, if not impossible, task. Consider the function

$$f(x) = e^{x^2}$$

An attempt to compute all of its derivatives leads to complicated derivatives. Instead, it is much easier to simply replace x with x^2 in Eq'n (5) leading to the expansion

$$e^{x^2} = \sum_{i=0}^{\infty} \frac{x^{2n}}{n!}.$$

EXERCISES 7.1

1. Find the first five nonzero terms of the Maclaurin series of the following functions:

$a)\ \frac{1}{1+2x}$ $b)\ \frac{1}{2-x}$ $c)\frac{2}{3x+4}$ $d)\ \frac{1}{1-x-x^2}$ $e)\ \frac{1}{1-2x^2}$ $f)\ \frac{1}{1+x-2x^2}$

$g)\ \frac{x-1}{1+9x^2}$ $h)\ \frac{x-1}{x^2-x-1}$ $i)\ \frac{1-x}{4+x^2}$ $j)\ \frac{1+3x}{1-x^3}$ $k)\ \frac{1-x+x^2}{1-x^2+x^4}$

$l)\ \frac{3-2x}{1-x+x^2-x^3}$ $m)\ e^x \sin x$

2. Using a computer, find the first eight terms of the Maclaurin series of the functions of Exercise 1.

3. Use the trigonometric formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

to prove that

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan -1\frac{1}{3}.$$

Use Eq'n (2) through the x^9 term and a calculator to estimate

$$4\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right).$$

Compare the resulting estimate of π with its actual value.

4. Use the trigonometric formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

to prove that

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan -1\frac{1}{239}.$$

Use Eq'n (2) through the x^9 term and a calculator to estimate

$$4\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{239}\right).$$

Compare the resulting estimate of π with its actual value.

5. Use the methods of this chapter to show that the Maclaurin series of $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \ldots.$$

6. Use at least two different methods to find the Maclaurin series of $\cos x$.

7. Let r be any nonzero rational number.

a) Show that the function $y = (1+x)^r$ satisfies the differential equation

$$(1+x)y' = ry.$$

b) Use the methods of this chapter to prove that the coefficients of the Maclaurin series of this function staisfy the equations

$$c_{k+1} = \frac{r-k}{k+1} \cdot c_k, \quad k = 0, 1, 2, \ldots.$$

c) Conclude that the Maclaurin series of $y = (1+x)^r$ is

$$(1+x)^r = 1 + \binom{r}{1} x + \binom{r}{2} x^2 + \binom{r}{3} x^3 + \ldots$$

where

$$\binom{r}{k} = \frac{r(r-1)\ldots(r-k+1)}{k!}, \quad k = 1, 2, 3, \ldots.$$

d) Explain the relationship between this Maclaurin series and the Binomial Theorem.

8. Let $f(x)$ be the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

a. Prove that

$$f^{(n)}(0) = 0 \quad \text{for each integer } n = 0, 1, 2, \ldots$$

b. Describe the McLaurin series of this function and explain why it converges for every value of x.

c. Conclude that the McLauring series of the function $f(x)$ converges for every $x \neq 0$ to a value that is different from $f(x)$.

2 Taylor's Theorem in One Variable

The most obvious advantage of series expansions is that they facilitate the (approximate) computation of many expressions that would otherwise be very awkward to compute. All that needs be done to get a good approximation of $\sqrt{e} = e^{1/2}$ is to compute an initial segment of the Maclaurin series of e^x for $x = 1/2$, say, for $n = 6$,

$$\sum_{i=0}^{6} \frac{(1/2)^n}{n!} = 1 + \frac{1}{2} + \frac{1}{2^2 \cdot 2!} + \frac{1}{2^3 \cdot 3!} + \frac{1}{2^4 \cdot 4!} + \frac{1}{2^5 \cdot 5!} + \frac{1}{2^6 \cdot 6!}$$

$$= \frac{75973}{46080} \approx 1.6487196...$$

which, our calculators tell us, is accurate to 5 decimal places. This brings us to a general issue with approximations. No purported approximation is of much use unless we have some idea of how good it is. In other words, every approximation must be accompanied by some upper bound on its deviation from the true value of the quantity that is being approximated. For this purpose Taylor offered the following theorem.

Theorem 2.1 *If the function* f(x) *has continuous derivatives up to the* (n+1)'th *order in the interior of the interval* I, *and* a *is in the interior of* I *then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n$$

where the remainder R_n *is given by*

$$R_n = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

PROOF: Note that the variables x and a are independent of each other. Define

$$R_n = R_n(x,a) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

It is clear that

$$R_n(x,x) = f(x) - f(x) = 0.$$

Moreover, the differentiation of R_n with respect to a yields, by the product rule,

$$\frac{\partial R_n}{\partial a}(x,a) = -\sum_{k=0}^{n}\left[\frac{f^{(k+1)}(a)}{k!}(x-a)^k - \frac{f^{(k)}(a)}{k!}k(x-a)^{k-1}\right]$$

which sum collapses to

$$-\left[\frac{f^{(n+1)}(a)}{n!}(x-a)^n - 0\right].$$

By the Fundamental Theorem of Calculus

$$R_n(x,a) = R_n(x,a) - R_n(x,x) = \int_x^a \frac{\partial R_n}{\partial t}(x,t)dt$$

$$= \int_a^x -\frac{\partial R_n}{\partial t}(x,t)dt = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

Q.E.D.

Lemma 2.2 *Let $f, g : [a, b] \to \Re$ be continuous functions with $g(x) \geq 0$ then there exists a number $\eta \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = f(\eta)\int_a^b g(x)dx$$

PROOF: Let m and M be the respective minimum and maximum values of $f(x)$ in $[a, b]$. Since $g(x) \geq 0$ it follows that

$$m\int_a^b g(x)dx = \int_a^b mg(x)dx \leq \int_a^b f(x)g(x)dx$$
$$\leq \int_a^b Mg(x)dx = M\int_a^b g(x)dx.$$

Consequently there exists a number μ between m and M such that

$$\mu\int_a^b g(x)dx = \int_a^b f(x)g(x)dx.$$

Since

$$m \leq \mu \leq M$$

it follows from the Intermediate Value Theorem that there exists a number $\eta \in [a, b]$ such that

$$f(\eta) = \mu$$

and hence

$$\int_a^b f(x)g(x)dx = f(\eta)\int_a^b g(x)dx.$$

Q.E.D.

Proposition 2.3 (Lagrange) *If R_n is the remainder term of Theorem yyy, then there exists a number $\theta \in [a - x, a + x]$ such that*

$$R_n(x, a) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\theta)$$

PROOF: This follows from an application of the lemma above to the preceding theorem. Q.E.D.

Corollary 2.4 *If R_n is the remainder term of Theorem yyy, and M is a positive number such that*

$$|f^{(n+1)}(\theta)| \leq M \quad \textit{for all} \quad \theta \in [a - x, a + x]$$

then

$$|R_n(x, a)| \leq \frac{M(x-a)^{n+1}}{(n+1)!}.$$

Example 2.5

In this section's first example we offered a rational estimate for the value of $\sqrt{e} = e^{1/2}$. It is now possible to add an estimate for the difference between this estimate and the real value. We begin with $a = 0, x = 1/2$ and the reasonable upper bound of 4. Since

$$|f^{(n+1)}(\theta)| \leq e^{\theta} \leq e^{1/2} \leq \sqrt{4} = 2,$$

we may set $M = 2$ and it follows that

$$|R_n| \leq \frac{2(1/2)^7}{7!} = \frac{1}{322560} \approx 3.1 \cdot 10^{-6}.$$

The actual value, correct to seven decimal places is 1.6487213. Note that the difference between this actual value and the estimate is

$$1.6487213 - 1.6487196 \approx 1.7 \cdot 10^{-6}$$

which is indeed smaller than $3 \cdot 10^{-6}$ which is the Lagrange estimate of the remainder.

It should be pointed out that the remainder estimates make it possible to give some of the formulas of the previous chapter a firmer logical grounding.

Example 2.6 *Prove that*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all real } x.$$

PROOF: Fix x and note that since the derivatives of $\sin x$ are all either $\pm \sin x$ or $\pm \cos x$, we can use $M = 1$ and so

$$|R_n(x, 0)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

However,

$$\lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Hence

$$\lim_{n\to\infty} R_n(x, 0) = 0$$

From which the desired result follows immediately. Q.E.D.

The Taylor polynomials of a function $f(x)$ are polynomials that approximate $f(x)$. For example the first degree Taylor polynomial of $y = e^x$ is

$$y = 1 + x$$

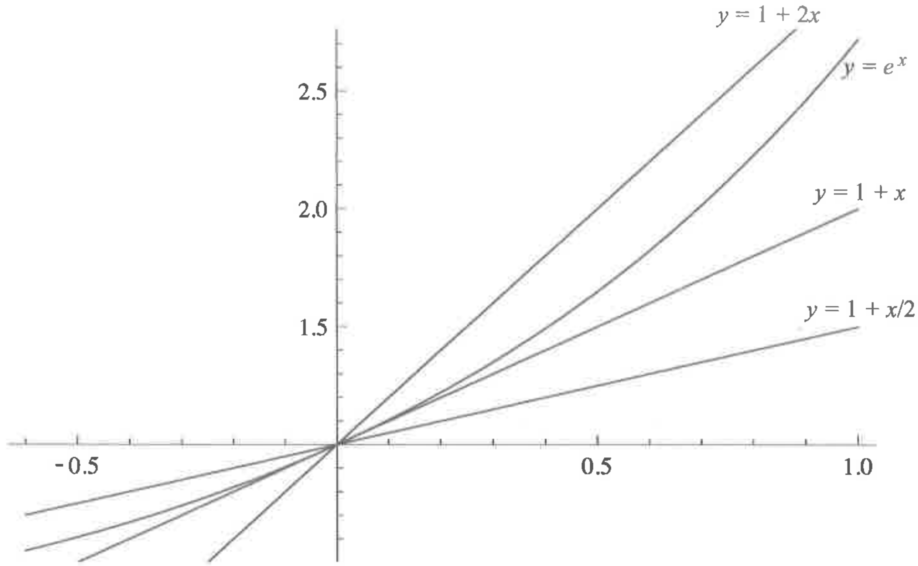

Figure 1: Linear approximations to the exponential curve

whose graph is the straight line tangent to $y = e^x$ at (0, 1). Now every straight line $y = 1 + mx$ through (0, 1) also approximates the exponential curve in the sense that

$$\lim_{x \to 0 e^x} -(1 + mx) = 0.$$

What distinguishes the tangent line $y = 1 + x$ from all of the others is that it provides the best approximation. This is visually clear from Figure xxx and can be algebraically demonstrated as follows. Because $1 + x$ is a Taylor polynomial it follows from Corollary 2.4 that for x sufficiently close to 0 (i.e., x such that $-2 < e^x < 2$)

$$|1 + x - e^x| = |R_1(x, 0)| < \frac{1}{2}x^2 \cdot 2 = x^2.$$

However, for each other line $y = 1 + mx, m \neq 1$ and for sufficiently small x,

$$|1 + mx - e^x| = |(m-1)x - (e^x - 1 - x)|$$
$$\geq |(m-1)x| - |(e^x - 1 - x)| = |(m-1)x)| - x^2$$
$$> \frac{|m-1|}{2}x >> x^2 = R_1(x, 0).$$

These considerations motivate a definition. The function $f(x)$ is said to *vanish to order n* at a provided

$$f(a) = f'(a) = f"(a) = \ldots = f^{(n)}(a) = 0.$$

For example, x^3 vanishes to order 2 at 0, $(x-3)^5$ vanishes to order 4 at 3, $x - \sin x$ vanishes to order 2 at 0, $\sin^3 x$ vanishes to order 2 at integer multiples of π, and $(x^2-1)^n$ vanishes to order n at $x = \pm 1$. Two functions $f(x)$ and $g(x)$ are said to have *contact of order* n at a if their difference

$$f(x) - g(x)$$

vanishes to order n at a. The polynomial functions

$$2 + x - 3x^2 + 7x^4 - x^6 \quad \text{and} \quad 2 + x - 3x^2 - x^5 - x^6$$

have order of contact 3. The functions $\sin x$ and x have order of contact 2.

Theorem 2.7 *Let $f : D \subset \Re \to \Re$ have $n+1$ continuous derivatives in some interval. Then the Taylor polynomial*

$$T_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the unique polynomial of degree at most n that has contact of order at least n with f(x) at a.

PROOF: It is clear that the $T_n(x,a)$ has the required contact with $f(x)$. Conversely, let $T_n'(x,a)$ be any polynomial of degree at most n that also has contact of order at least n at a. Hence

$$T_n(x,a) - T_n'(x,a)$$

vanishes to order n at a. It follows that the polynomials T and T' have identical coefficients and so they must be equal. Q.E.D

EXERCISES 7.2

1. Use the remainder estimates to prove that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for } -1 < x < 1.$$

2. Use the remainder estimates to prove that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all real } x.$$

3. Use the remainder estimates to prove that for every rational number r

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \quad \text{for all } \quad -1 < x < 1.$$

4. Use the methods of this section to obtain a rational estimate of e that comes to within 10^{-5} of its true value. (Go ahead and assume that $e < 3$).

5. Use the methods of this section to obtain a rational estimate of $\sqrt{1.6}$ that comes to within 10^{-5} of its true value.

6. Use the methods of this section to obtain a rational estimate of $1.5^{5/7}$ that comes to within 10^{-5} of its true value.

7. Use the methods of this chapter to obtain a rational estimate of $\ln 1.5$ that comes within 10^{-5} of its true value.

8. Use the remainder estimates to prove that for all x such that -1 $< x < 1$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}.$$

3 Taylor Polynomials in Two Variables

The informal methodology that was used in the beginning of this chapter to introduce the theory of Taylor series applies to functions of two, or more, variables as well. For example, by Eq'n (1) and the Binomial Theorem,

$$\frac{1}{1-x-y} = \frac{1}{1-(x+y)} = 1 + x + y + (x+y)^2 + (x+y)^3 + \ldots$$

$$= \sum_{m,n=0}^{\infty} \binom{m+n}{m} x^m y^n, \quad \text{provided } |x| + |y| < 1. \tag{10}$$

Similarly,

$$\cos(x+y) = \sum_{n=0}^{\infty} (-1)^n \frac{(x+y)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \frac{(-1)^n \binom{2n}{m}}{(2n)!} x^m y^{2n-m}$$

The differentiation process that yielded the coefficients of the Taylor series in Section 1 can also be applied to functions of two variables. If the coefficient of the term $(x-a)^m(y-b)^n$ in the Taylor series, assuming there is such a series, is denoted by $c_{m,n}$, then

$$f(x,y) = \sum_{m,n=0}^{\infty} c_{m,n}(x-a)^m(y-b)^n.$$

To derive the value of a specific coefficient $c_{m,n}$ we first differentiate this equation m times with respect to x and n times with respect to y. If we substitute $x = a, y = b$ we obtain

$$\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(a,b) = c_{m,n} m! n!$$

and hence

$$c_{m,n} = \frac{1}{m!n!} \frac{\partial^{m+n} f}{\partial x^m \partial y^n}(a,b). \tag{11}$$

Example 3.1 *Use Eq'n (11) to find the coefficients of the Taylor series of*

$$f(x,y) = \frac{1}{1-x-y}.$$

Straightforward applications of the chain rule to $(1-x-y)^{-1}$ yield

$$\frac{\partial^m f}{\partial x^m} = m!(1-x-y)^{-m}$$

and

$$\frac{\partial^n}{\partial y^n}\left(\frac{\partial^m f}{\partial x^m}\right) = \frac{\partial^{m+n} f}{\partial x^m \partial y^n}$$
$$= m!(m+1)(m+2)\ldots(m+n)(1-x-y)^{-m-n}$$
$$= \frac{(m+n)!}{(1-x-y)^{m+n}}.$$

It follows that for $(a,b) = (0,0)$ we have

$$c_{m,n} = \frac{1}{m!n!} \cdot (m+n)!(1-0-0)^{-m-n}$$
$$= \frac{(m+n)!}{m!n!} = \binom{m+n}{m}$$

which agrees with (10).

The Taylor polynomial of degree n of the function $f(x,y)$ is the finite partial sum

$$T_n(a,b) = \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \frac{\partial^m f(a,b)}{\partial x^k \partial y^{m-k}} (x-a)^k (y-b)^{m-k}. \tag{12}$$

For example, the Taylor polynomials of degrees $n = 0, 1, 2, 3$ of the function

$$f(x,y) = \frac{1}{1-x-y}$$

are, respectively,

$$T_0(x,y) = 1,$$
$$T_1(x,y) = 1 + x + y,$$

$$T_2(x,y) = 1 + x + y + (x+y)^2,$$

and

$$T_3(x,y) = 1 + x + y + (x+y)^2 + (x+y)^3.$$

This can be verified either by examining Eq'n (12) or by computing all the necessary derivatives. As happened in the single variable case, such polynomials are often computed without resorting to the formula of Eq'n (12).

Example 3.2 *Compute the Taylor polynomial $T_3(0,0)$ for the function*

$$f(x,y) = e^x + \sin y.$$

It follows from the previous section that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

and

$$\sin y = y - \frac{y^3}{6} + \ldots$$

Hence, for $f(x,y) = e^x + \sin y$

$$T_3(0,0) = 1 + x + y + \frac{x^2}{2} + \frac{x^3}{6} + \frac{y^3}{6} + \ldots$$

Example 3.3 *Compute the Taylor polynomial $T_3(0,0)$ for the function*

$$f(x,y) = e^x \sin y.$$

Here

$$e^x \sin y = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots)(y - \frac{y^3}{6} + \ldots)$$

and hence

$$T_3(0,0) = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \ldots$$

Example 3.4 *Compute the Taylor polynomial $T_3(0,0)$ for the function*

$$f(x,y) = \frac{e^x}{1 + \sin y}.$$

Here

$$f(x,y) = e^x(1 - \sin y + \sin^2 y - \sin^3 y + \ldots)$$

$$= (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots)\times$$

$$[1 - (y - \frac{y^3}{6} + \ldots) + (y - \frac{y^3}{6} + \ldots)^2 - (y - \frac{y^3}{6} + \ldots)^3 + \ldots$$

$$= (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots) \times (1 - y + y^2 - \frac{5y^3}{6})$$

$$= 1 + x - y + \frac{x^2}{2} - xy + y^2 + \frac{x^3}{6} - \frac{x^2 y}{2} + xy^2 - \frac{5y^3}{6} + \ldots$$

and so

$$T_3(0,0) = 1 + x - y + \frac{x^2}{2} - xy + y^2 + \frac{x^3}{6} - \frac{x^2 y}{2} + xy^2 - \frac{5y^3}{6}.$$

There is a simple device that greatly facilitates the rigorous discussion of Taylor series of two variables by reducing them to the series of a function of a single variable. We demonstrate this procedure with a simple case where $(a, b) = (0, 0)$ and $n = 2$ which will turn out to be very useful in the next chapter in a very general setting. Let $f(x, y)$ be a function with continuous partial third derivatives. We define an auxiliary function

$$F(t) = f(xt, yt)$$

and note that $F(0) = f(0,0)$ and $F(1) = f(x, y)$. By the results of the previous section we have

$$F(t) = F(0) + F'(0)t + \frac{1}{2}F''(0)t^2 + R_2(t, 0).$$

However, by the chain rule

$$F'(t) = f_1(xt, yt)x + f_2(xt, yt)y$$

$$F''(t) = f_{11}(xt, yt)x^2 + 2f_{12}(xt, yt)xy + f_{22}(xt, yt)y^2$$

so that

$$f(x, y) = f(0,0) + f_1(0,0)x + f_2(0,0)y$$

$$+ f_{11}(0,0)x^2 + \frac{1}{2}f(0,0)xy + f_{22}(0,0)y^2 + R_2(1,0).$$

Here

$$R_2(1,0)$$

$$= \frac{1}{3!}\{x^3 f_{111}(\theta x, \theta y) + 3x^2 y f_{112}(\theta x, \theta y) + 3xy^2 f_{122}(\theta x, \theta y) + y^3 f_{222}(\theta x, \theta y)\}$$

Hence, for some M

$$|R_2(1,0)| \leq \frac{M}{3!}(|x| + |y|)^3 \qquad (13)$$

Let (a, b) be a fixed point in the domain of the real valued function $f(x, y)$ which is assumed to be infinitely??? differentiable. Define a new function

$$F(t) = f(a + (x - a)t, b + (y - b)t).$$

Note that

$$F(0) = f(a, b) \quad \text{and} \quad F(1) = f(x, y).$$

It follows from repeated applications of the chain rule that

$$F'(t) = (x-a)f_x + (y-b)f_y$$

$$F''(t) = (x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}$$

$$\cdots$$

$$F^{(n)}(t) = (x-a)^n f_{x^n} + \binom{n}{1}(x-a)^{n-1}(y-b)f_{x^{n-1}y}$$

$$+\binom{n}{2}(x-a)^{n-2}(y-b)^2 f_{x^{n-2}y^2} + \ldots + (y-b)^n f_{y^n}$$

$$\cdots$$

or, using symbolic notation,

$$F^{(n)}(t) = [(x-a)f_x + (y-b)f_y]^{(n)}.$$

Theorem 3.5 *Let $f : D \subset \Re^2 \to \Re$ which is infinitely differentiable. Then*

$$f(x,y) = f(a,b) + \{(x-a))f_x(a,b) + (y-b)f_y(a,b)\}$$

$$+\frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\}$$

$$+\ \ldots$$

$$+\frac{1}{n!}\left\{(x-a)^n f_{x^n}(a,b) + \binom{n}{1}(x-a)^{n-1}(y-b)f_{x^{n-1}y}(a,b)\right.$$

$$\left. + \ldots + (y-b)^n f_{y^n}(a,b)\right\}$$

$$+R_n$$

where, for some $0 < \theta < 1$,

$$R_n = \frac{1}{(n+1)!}\{(x-a)f_x[a+\theta(x-a), b+\theta(y-b)$$

$$+ (y-b)f_y[a+\theta(x-a), b+\theta(y-b)]\}^{(n+1)}$$

PROOF: Apply Taylor's formula with Lagrange's form of the remainder to the function $F(t)$ and then set $t = 1$. Q.E.D.

Theorem 3.6 (Alternative form)

$$f(x,y) = \sum_{k=0}^{n} \frac{1}{k!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)]^{(k)}.$$

Theorem 3.7 (Alternative form)

$$f(x,y) = \sum_{k=0}^{n} \frac{1}{k!}[(x-a)f_x + (y-b)f_y]^{(k)}(a,b) + R_n$$

where

$$R_n = \frac{1}{(n+1)!}[(x-a)f_x+(y-b)f_y]^{(n+1)}(a+\theta(x-a), b+\theta(y-b)), \quad 0<\theta<1.$$

Note that "in general"

$$R_n = o\{(\sqrt{(x-a)^2+(y-b)^2})^n\} \quad \text{as } (x,y) \to (a,b)$$

EXERCISES 7.3

1. Find the Maclaurin series of e^{x+y} in two ways.

2. Find the Maclaurin series of $\sin(x+y)$ in two ways.

3. Find the Maclaurin series of $\cos(x+y)$ in two ways.

4. If r is a rational number, find the Maclaurin series of $(1+x+y)^r$ in two ways.

5. Find the Taylor polynomial of degree 3 for each of the following functions at $(a=b=0)$.

a. $\tan(x+y)$ b. $\sin(y\cos x)$ c. $\cos(y+\cos x)$ d. $\dfrac{e^x}{\cos y}$ e. $\dfrac{e^x}{1+\cos y}$

4 Trigonometric series?

EXERCISES 7.4