8. CONICS AND EXTREMA
September 16, 2009

1 Introduction

The *Conics* are the curves that are obtained by intersecting a double cone or a circular cylinder with a plane. They consist of ellipses, parabolas, hyperbolas, pairs of coplanar straight lines, single straight lines, single points, and the empty set (see Figures 1 and 2). They were defined and investigated by the Greeks over 2300 years ago and Apollonius of Perga (ca. 262 - ca. 190 B.C.) proved several hundreds of theoretical propositions regarding them. Johannes Kepler (1571 - 1630) observed that planets and comets revolve in elliptical orbits about the sun and subsequently many other scientific applications of the conic sections have been discovered. In this chapter it is proved that the conic sections constitute all the graphs of the general second degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F. \tag{1}$$

This theorem has been attributed to Pierre Fermat (1601? - 1665). It is then applied to the topic of determining the maxima and minima of functions in two variables.

2 Translations and Rotations of Cartesian Coordinate Systems

Coordinate systems possess a degree of arbitrariness in that the origin can be placed anywhere in addition to some more freedom in placing the coordinate axes. It is not surprising then that different coordinate systems will yield different algebraic descriptions of the same geometrical object. For example, the point A in Figure 3 has positive coordinates in the coordinate system (O, x, y) whereas its coordinates relative to the system (O, x', y') are both

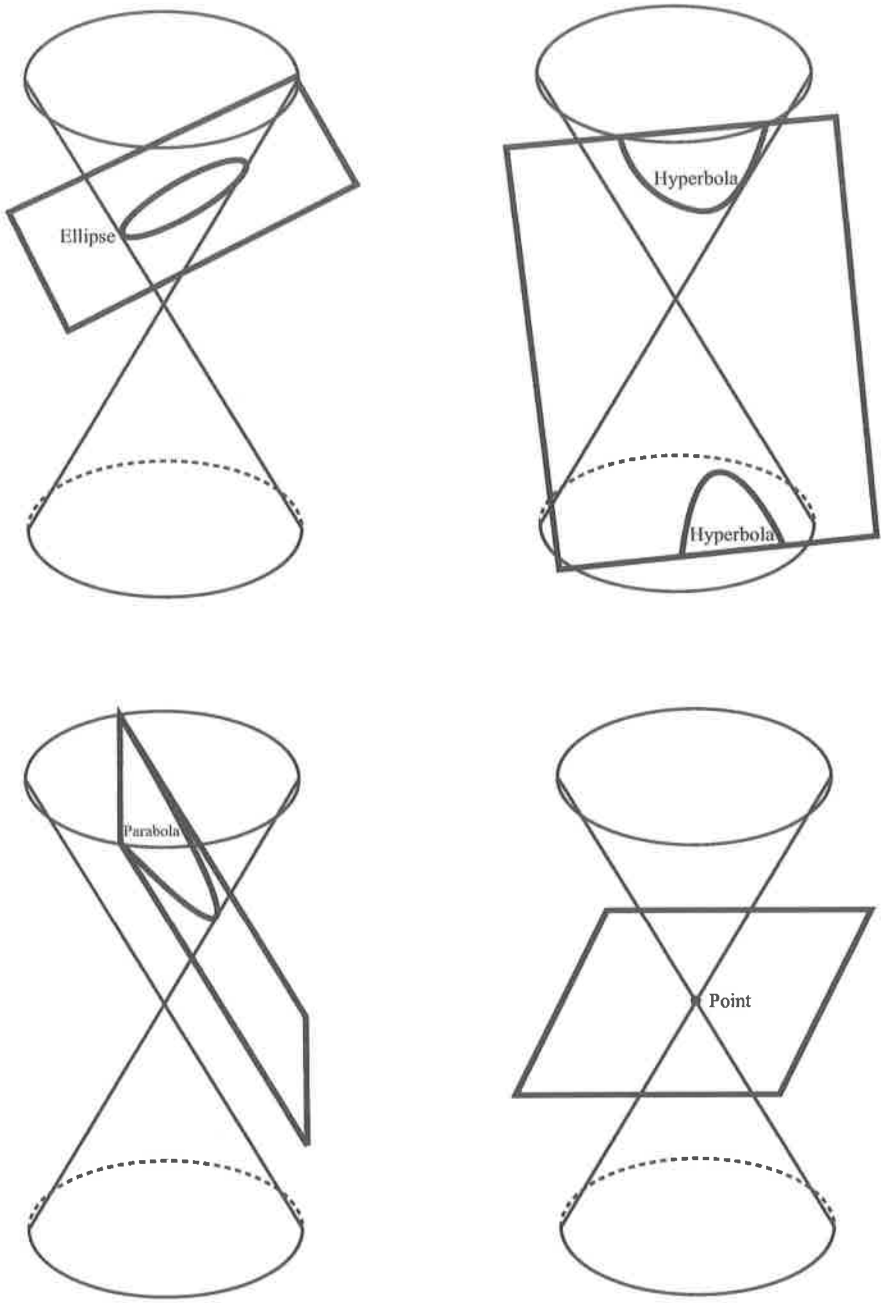

Figure 1: The intersections of a plane with a cone

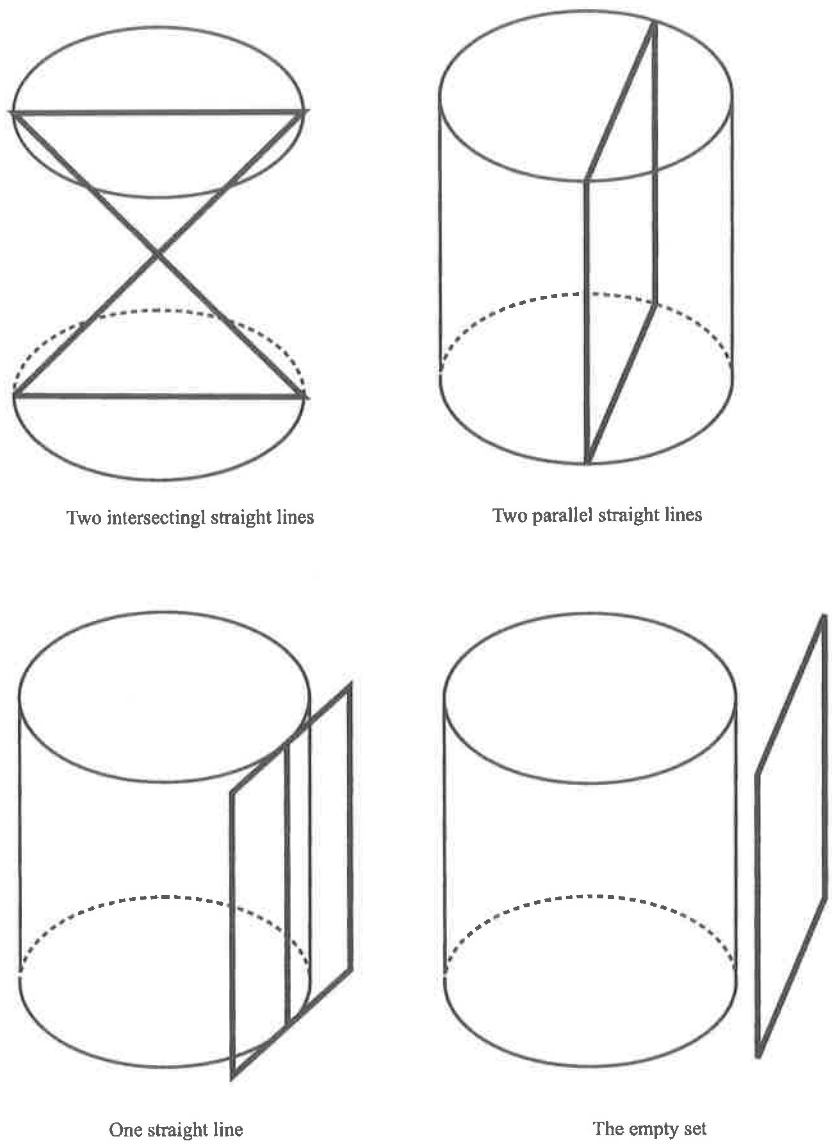

Figure 2: The intersections of a plane with a cylinder

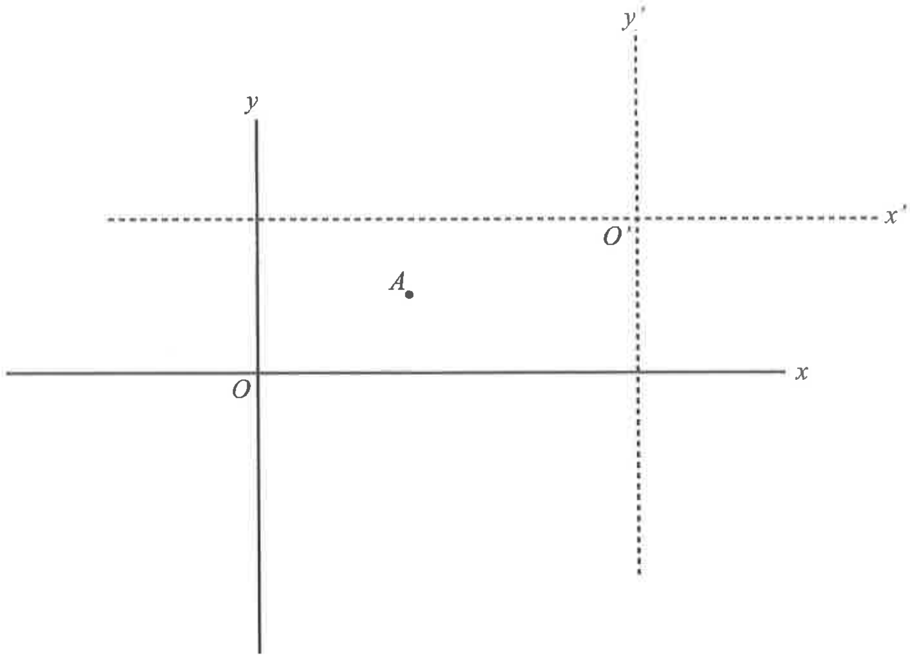

Figure 3: Different coordinates

negative. We shall see that this flexibilty is actually very useful as it makes it possible to find optimal coordinate systems that simplify the algebraic equations that arise.

We first examine the effect that changing the origin without changing the (directions of the) axes has. Suppose a coordinate system (O, x, y) is given, and a point O', with coordinates (h, k) is selected as the origin of the new coordinate system (O', x', y') whose axes are parallel to the old axes. More precisely, the new coordinate system is obtained by a parallel translation of the old system (Fig. 4). The parallel translation that moves the origin to the point (h, k) is denoted by $\tau_{(h,k)}$. It is clear from this figure that the relation between the old and new coordinates is given by

$$x = x' + h, \quad y = y' + k. \tag{2}$$

In other words, the arbitrary point A whose coordinates in the old coordinate system are (x, y) has coordinates $(x', y') = (x - h, y - k)$ in the new system.

Example 2.1 *Find the coordinates of the point (3, -2) relative to the translated coordinate system with origin at (-1, 3).*

The old coordinates of O' were $(x, y) = (-1, 3)$ whereas its new coordinates are $(x', y') = (0, 0)$. It follows from Eq'n (2) that

$$-1 = 0 + h, 3 = 0 + k \qquad \text{or} \qquad h = -1, k = 3.$$

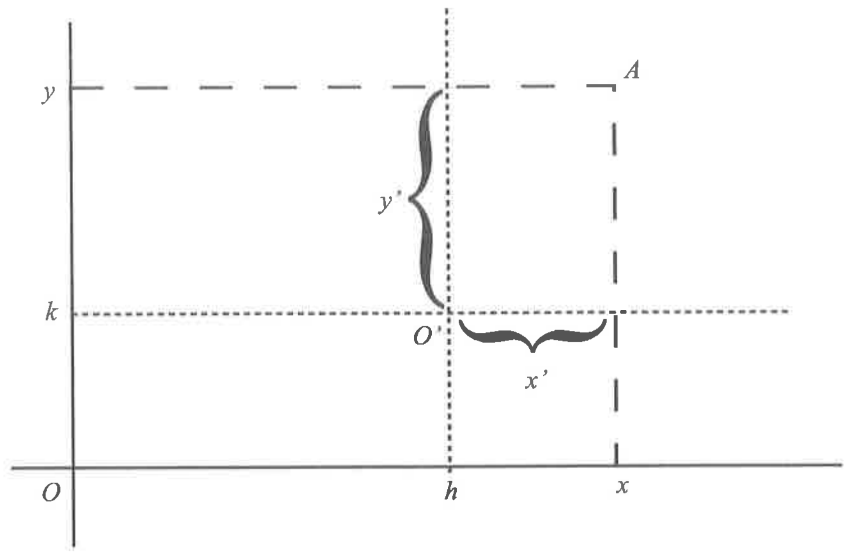

Figure 4: Translation of a coordinate system

Hence the new coordinates of (3, -2) are

$$x' = 3 - (-1) = 4, y' = -2 - 3 = -5 \qquad \text{or} \qquad (x', y') = (4, -5).$$

Example 2.2 *Find the equation of the straight line $3x + 4y = 5$ after the coordinate system is subjected to the translation $\tau_{(2,-1)}$.*

Substitute Eq'n (2) into $3x + 4y = 5$ to obtain

$$3(x' + 2) + 4(x' + (-1)) = 5$$

or

$$3x' + 4y' = 3.$$

Example 2.3 *Find the equation of the circle of radius 5 and center (-2,3) after the coordinate system is subjected to the translation $\tau_{(2,-1)}$*

The circle's equation in the old coordinate system is $(x+2)^2 + (y-3)^2 = 25$. The substitution of Eq'n (2) into this equation yields

$$(x' + 2 + 2)^2 + (y' - 1 - 3)^2 = 25$$

or

$$(x' + 4)^2 + (y' - 4)^2 = 25.$$

Next we examine the effect that changing the axes without changing either their relative disposition or the origin has on the coordinates. In other words, the new coordinate system is obtained from the old one by a rotation

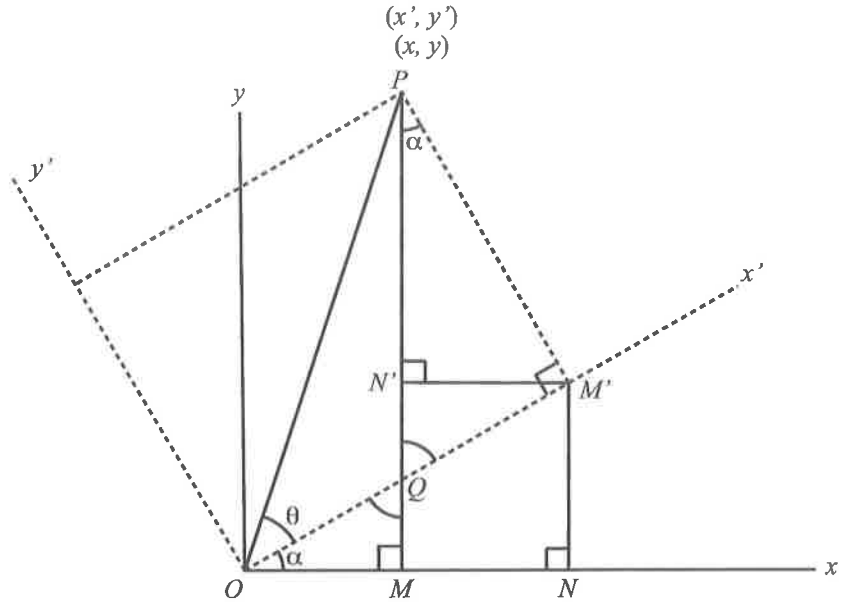

Figure 5: Rotation of a coordinate system

about O (Fig. 5) by an angle of, say, α. This rotation is denoted by R_α. The comparison of the angles of ΔOQM and $\Delta PQM'$ allows us to conclude that

$$\angle M'PQ = \angle MOQ = \alpha.$$

Hence,

$$\begin{aligned} x = OM &= ON - MN = ON - M'N' \\ &= OM' \cos\alpha - PM' \sin\alpha = x' \cos\alpha - y' \sin\alpha. \end{aligned}$$

Thus,

$$x = x' \cos\alpha - y' \sin\alpha \tag{3}$$

and similarly (Exercise 1)

$$y = x' \sin\alpha + y' \cos\alpha. \tag{4}$$

When Eq'ns (3,4) are solved simultaneously for x' and y' we obtain

$$x' = x\cos\alpha + y\sin\alpha \tag{5}$$

$$y' = -x\sin\alpha + y\cos\alpha. \tag{6}$$

Example 2.4 *Find the new coordinates of the points* (1, 0) *and* (1, 1) *if the coordinate axes are subjected to the rotation* R_{60°.

The new coordinates of (1, 0) are given by Eq'ns (5,6):

$$x' = 1 \cdot \cos 60° + 0 \cdot \sin 60° = \tfrac{1}{2}$$
$$y' = -1 \cdot \sin 60° + 0 \cdot \cos 60° = -\tfrac{\sqrt{3}}{2}.$$

Similarly, the new equations of (1,1) are

$$x' = 1 \cdot \cos 60° + 1 \cdot \sin 60° = \tfrac{1+\sqrt{3}}{2}$$
$$y' = -1 \cdot \sin 60° + 1 \cdot \cos 60° = \tfrac{1-\sqrt{3}}{2}.$$

Example 2.5 *Find the new equation of the straight line* 3x + 4y = 5 *if the coordinate axes are subjected to the rotation* $R_{60°}$.

Substitute Equations (3, 4) into the given linear equation to obtain

$$3(x' \cos 60° - y' \sin 60°) + 4(x' \sin 60° + y' \cos 60°) = 5$$

or

$$\left(\frac{3}{2} + 2\sqrt{3}\right) x' + \left(2 - \frac{3\sqrt{3}}{2}\right) y' = 5.$$

EXERCISES 8.2

1. Find the coordinates of the points (2, -7), (-2, 3), and (a, b) relative to the translated coordinate system with origin at $(1, -3)$.
2. Find the coordinates of the points (2, -7), (-2, 3), and (a, b) relative to the translated coordinate system with origin at $(-1, 3)$.
3. Find the coordinates of the points (2, -7), (-2, 3), and (a, b) relative to the translated coordinate system with origin at (c, d).
4. Find the equation of the straight line $2x - 3y + 1 = 0$ after the coordinate system is subjected to the translation $\tau_{(-1,3)}$.
5. Find the equation of the straight line $Ax + By = C$ after the coordinate system is subjected to the translation $\tau_{(a,b)}$.
6. Find the equation of the circle $x^2 + y^2 - 4x + 6y = 14$ after the coordinate system is subjected to the translation $\tau_{(2,-3)}$.
7. Find the equation of the parabola $y^2 = 6x$ after the coordinate system is subjected to the translation $\tau_{(2,-3)}$.
8. Find the new coordinates of the points ((1, 0), (0, 1), (1, 1) if the coordinate axes are subjected to the rotation R_α where α equals
a) 30° b) 45° c) 60° d) 90° e) 120° f)135° g) 180° h) 270°.
9. Find the new coordinates of the points ((1, 2), (-1, 1), (-3,-4) if the coordinate axes are subjected to the rotation R_α where α equals
a) 30° b) 45° c) 60° d) 90° e) 120° f)135° g) 180° h) 270°.

10. Find the new equation of the straight line $y = x$ if the coordinate axes are subjected to the rotation R_α where α equals

a) 30° b) 45° c) 60° d) 90° e) 120° f)135° g) 180° h) 270°.

11. Find the new equation of the straight line $y = x$ if the coordinate axes are subjected to the rotation R_α.

12. Find the new equation of the straight line $x + 2y = 3$ if the coordinate axes are subjected to the rotation $R_\alpha$where α equals

a) 30° b) 45° c) 60° d) 90° e) 120° f)135° g) 180° h) 270°.

13. Find the new equation of the straight line $Ax + By = C$ if the coordinate axes are subjected to the rotation R_α.

14. Find a new equation of the graph of $y = 1/x$ if the coordinate system is subjected to the rotation R_{45°.

15. Find a new equation of the graph of $y = x^2$ if the coordinate system is subjected to the rotation R_{45°

3 Graphs of Second Degree Equations

The most general linear equation in two unknowns is of course

$$Ax + By + C = 0.$$

where A, B, and C are not all zero. Its graph is always a straight line, unless $A = B = 0$ in which case the graph is empty. It is natural to look for the graphs of second degree equations as well. Examples of these are

$$x^2 + 4y^2 = 1, \quad \text{an ellipse} \tag{7}$$

$$x^2 - 4y^2 = 1, \quad \text{a hyperbola} \tag{8}$$

$$y = x^2, \quad \text{a parabola} \tag{9}$$

$$y^2 = x^2, \quad \text{two intersecting straight lines} \tag{10}$$

$$x^2 - x = 0, \quad \text{two parallel straight lines} \tag{11}$$

$$x^2 + y^2 = 0, \quad \text{a point} \tag{12}$$

$$x^2 + y^2 = -1, \quad \text{the null set} \tag{13}$$

$$x + y = 0, \quad \text{a straight line} \tag{14}$$

The most general equation of this degree is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F. \tag{15}$$

where A, B, C, D, E, and F are not all zero. We shall show that a second degree equation is always of one of the types mentioned in Eq'ns (7-14).

Let G denote the graph of Eq'n (1). We first find a rotation of the coordinate system such that the equation of G in the new system has no "mixed" xy terms. Of course, if $B = 0$ there is nothing to do and so we assume that $B \neq 0$. To determine the appropriate angle of rotation α we substitute Eq'ns (3,4) into the general quadratic (15) to obtain

$$A(x'\cos\alpha - y'\sin\alpha)^2 + B(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha)$$
$$+C(x'\sin\alpha + y'\cos\alpha)^2 + D(x'\cos\alpha - y'\sin\alpha) +$$
$$E(x'\sin\alpha + y'\cos\alpha) + F = 0.$$

Since the purpose of this rotation is to nullify the $x'y'$ coefficient we must chose an α such that

$$B(\cos^2\alpha - \sin^2\alpha) + 2(C - A)\sin\alpha\cos\alpha = 0$$

or

$$\cot 2\alpha = \frac{A - C}{B}.$$

Since $B \neq 0$ and the range of the cotangent function is the whole of $\Re$, such an α always exists. Thus we may, and do, assume that $B = 0$.

If $A = C = 0$ Eq'n (16) reduces to

$$Dx + Ey = F$$

which is the equation of a straight line.

If $A = 0$ but $C \neq 0$ then

$$Cy^2 + Dx + Ey = F$$

whose graph is a parabola if $D \neq 0$ and one or two straight lines if $D = 0$. Such too is the case if $A \neq 0$ but $C = 0$.

If both $A \neq 0$ and $C \neq 0$ are numbers of the same sign, then the graph is an ellipse (or a point or the null set). When A and C have opposite signs the graph is a hyperbola (or a pair of intersecting lines).

Thus we have proved the following theorem.

Theorem 3.1 *The graph of an equation of the form*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F$$

is one the following: an ellipse, a hyperbola, a parabola, two intersecting straight lines, two parallel straight lines, one straight line, a point, or the null set.

□

When studying the effect of transformations it is useful to have *invariants* which are expressions whose form is unchanged by the transformations. The next proposition demonstrates the invariance of two such expressions for the conics: $B^2 - 4AC$ and $A + C$.

Proposition 3.2 *Suppose the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F$$

is transformed into the equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' = F'$$

by means of either a translation $\tau_{(h,k)}$ or a rotation R_α (of coordinates). Then

$$B'^2 - 4A'C' = B^2 - 4AC \quad \text{and} \quad A' + C' = A + C. \tag{16}$$

PROOF: We discuss the translations first. In this case, it follows from Exercise 1 that

$$A' = A, \quad B' = B, \quad C' = C.$$

Consequently both the equations of (16) are clearly satisfied.

Turning to rotations, we know from Exercise 2 that

$$A' + C' = A(\cos^2\alpha + \sin^2\alpha) + B \cdot 0 + C(\sin^2\alpha + \cos^2\alpha) = A + C.$$

The proof that $B'^2 - 4A'C' = B^2 - 4AC$ is equally straightforward though somewhat longer. The details are relegated to Exercise 3. Q.E.D.

The quantity $B^2 - 4AC$ is the *discriminant* of the general quadratic (15). The table below uses the invariance of the discriminant to summarize the classification of the conic sections. It is a corollary of the invariance of the discriminant.

$B^2 - 4AC$	Type	Degenerate cases
< 0	Ellipse	Point, null set
$= 0$	Parabola	One or two parallel straight lines
> 0	Hyperbola	Two intersecting straight lines

Example 3.3 *The graph of*

$$x^2 + 4xy + y^2 = 5$$

is a hyperbola because

$$B^2 - 4AC = 4^2 - 4 \cdot 1 \cdot 1 = 12 > 0.$$

In fact, when the coordinates are rotated so as to eliminate the mixed term, we obtain

$$A' + C' = A + C = 1 + 1 = 2$$

$$A'C' = \frac{0^2 - 4A'C'}{-4} = \frac{B^2 - 4AC}{-4} = -\frac{12}{4} = -3$$

Thus either $A' = 3, C' = -1$ or $A' = -1, C' = 3$, leading to either

$$3x'^2 - y'^2 = 5 \quad \text{or} \quad -x'^2 + 3y'^2 = 5.$$

Example 3.4 *The graph of*

$$4x^2 + 4xy + y^2 = 5$$

is a parabola because

$$B^2 - 4AC = 4^2 - 4 \cdot 4 \cdot 1 = 0.$$

In fact, the equation can be factored as

$$(2x + y)^2 = 5 \quad \text{or} \quad 2x + y = \pm\sqrt{5}$$

whose graph consists of two parallel straight lines of slope -2.

EXERCISES 8.3

1. Suppose $B^2 - 4AC \neq 0$. Prove that if the coordinate system is subjected to the translation $\tau_{(h,k)}$ where

$$h = \frac{2CD - BE}{B^2 - 4AC}, \qquad k = \frac{2AE - BD}{B^2 - 4AC}$$

then the general quadratic of Equation (15) is transformed into

$$Ax'^2 + Bx'y' + Cy'^2 = F'.$$

2. Prove that if the coordinate system is subjected to a rotation R_α where

$$\alpha = \frac{1}{2}\cot^{-1}\frac{A - C}{B}$$

then the general quadratic equation (15) has the new equation

$$A'x'^2 + C'y'^2 + Dx' + Ey' = F'.$$

where

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$
$$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha$$
$$D' = D\cos\alpha + E\sin\alpha$$
$$E' = E\cos\alpha - D\sin\alpha$$
$$F' = F$$

3. Complete the missing details in the last paragraph of the proof of Proposition 3.2.

4. Let G be the graph of the equation

$$Ax^2 + Cy^2 = F.$$

Prove the following assertions:

a. If $AC > 0$ then G is an ellipse, the origin, or the null set;

b. If $AC < 0$ then G is either a hyperbola or a pair of intersecting straight lines;

c. If $AC = 0$ then G is the entire plane, a pair of parallel straight lines, a single straight line, or the empty set.

5. a) Prove that if $C \neq 0$ and $\neq 0$ then the graph of the equation

$$Cy^2 + Dx + Ey = F$$

is a parabola.

b) Explain why if $C \neq 0$ then the graph of the equation

$$Cy^2 + Ey = F$$

is either a straight line or a pair of parallel straight lines.

6. Determine the nature of the graphs of the following equations.

a. $x^2 + xy + y^2 = 1$;
b. $x^2 + xy - y^2 = 1$;
c. $x^2 - xy + y^2 = 1$;
d. $x^2 - xy - y^2 = 0$;
e. $x^2 + 2xy + y^2 = 1$;
f. $x^2 + 2xy - y^2 = 1$;
g. $3x^2 - 2xy + y^2 = 1$;
h. $-x^2 - 2xy - y^2 = 1$;
i. $4x^2 + 4xy + y^2 = 1$;
j. $4x^2 + 4xy - y^2 = 1$;
k. $4x^2 - 4xy + y^2 = 1$;
l. $-4x^2 + 4xy - 4y^2 = 1$.
m. $x^2 - xy - y^2 = -1$.
n. $x^2 - 2xy - 3y^2 = 0$.
o. $x^2 - 2xy - 3y^2 = -1$.

7. For which values of A, B, C, F is the graph of

$$Ax^2 + Bxy + Cy^2 = F$$

a(n)

a. hyperbola
b. parabola
c. ellipse
d. pair of parallel straight lines
e. pair of intersecting straight lines
f. single straight line
g. single point
h. empty set.

4 Quadratic Forms

A quadratic form is a function of the form

$$z = Q(x, y) = Ax^2 + Bxy + Cy^2, \qquad A, B, C \text{ not all zero.}$$

The form is said to be *positive definite* if it vanishes only at (0,0) and its range consists of the nonnegative reals; it is *negative definite* if its range consists of the non positive numbers. The forms

$$x^2 + y^2 \quad \text{and} \quad -x^2 - y^2$$

exhibit such behaviors. The form Q is *definite* if it is either positive or negative definite.

If the form Q assumes both positive and negative values, it is said to be indefinite. Such is the case for

$$Q(x, y) = x^2 - y^2.$$

Finally, if the form also vanishes at other points besides the origin, but otherwise its values are either all positive or all negative, then it is $semi-definite$. Such is the case for the forms

$$(x - y)^2 = x^2 - 2xy + y^2 \quad \text{and} \quad -(x - y)^2 = -x^2 + 2xy - y^2.$$

Lemma 4.1 *Let $Q(x, y)$ be a quadratic form. Then the range of Q is not affected by a transformation of the xy-plane.*

PROOF: This is obvious.

Lemma 4.2 *Let $Q(x, y) = Ax^2 + Bxy + Cy^2$ be a quadratic form. Then Q is*

a) definite if and only if $B^2 - 4AC < 0$ (positive if $A > 0$ and negative if $A < 0$),

b) indefinite if and only if $B^2 - 4AC > 0$.

PROOF: a) Note that

$$Q(x,y) = Ax^2 + Bxy + Cy^2$$
$$= A\left[\left(x + \frac{B}{2A}y\right)^2 + \frac{4AC - B^2}{4A^2}y^2\right] \tag{17}$$

Hence if $B^2 - 4AC < 0$ the bracketed term of (17) above is the sum of two squares and so Q is positive definite if $A < 0$ and negative definite if $A > 0$.

If $B^2 - 4AC > 0$ the bracketed term is the difference of two squares and hence, by choosing suitable values of x and y, it can assume both negative and positive values. Thus, Q is indefinite. Q.E.D.

Lemma 4.3 *Let $Q(x,y)$ be a positive definite quadratic form. Then there exists a positive number m such that for all $(x,y) \neq (0,0)$*

$$\frac{Q(x,y)}{x^2+y^2} \geq m.$$

PROOF: Assume first that $B = 0$. Then $0 > B^2 - 4AC = -4AC$, and hence A and C are both positive. Consequently

$$\frac{Ax^2 + Cy^2}{x^2+y^2} \geq \min\{A, C\} > 0.$$

If $B \neq 0$, then there is a rotation of the x and y axes such that

$$\frac{Ax^2 + Bxy + Cy^2}{x^2+y^2} = \frac{A'x'^2 + C'y'^2}{x'^2+y'^2} \geq \min\{A', C'\} > 0.$$

The reason A' and C' are both positive is that by Eq'n (18)

$$-A'C' = 0^2 - A'C' = B^2 - 4AC < 0.$$

Since this rotation does not alter the range of values of $Q(x,y)$, we are done. Q.E.D.

Lemma 4.4 *Let $Q(x,y)$ be any quadratic form. Then the function*

$$\frac{Q(x,y)}{x^2+y^2}$$

is constant along each straight line through the origin.

PROOF: Note that

$$Q(tx, ty) = t^2 Q(x,y).$$

Since the quadratic form $x^2 + y^2$ has the same property, the lemma follows when the common t^2 term is cancelled. Q.E.D.

EXERCISES 8.4

1. For each of the following quadratic forms decide whether it is definite, semi-definite, or indefinite and determine its sign.

a. $x^2 - 3xy + 2y^2$ b. $4x^2 - 2xy + 2y^2$ c. $x^2 - 3xy - 2y^2$

d. $x^2 + 3xy + 2y^2$ e. $-2x^2 + 2y^2$ f. $x^2 - 3xy$ g. $3xy + 2y^2$

h. $x^2 - 3xy + 2y^2$

2. Verify that $x^2 - 2xy + 3y^2$ is a positive definite form. Find the largest possible number m and the smallest number M such that such that for all $(x, y) \neq (0, 0)$

$$m \leq \frac{x^2 - 2xy + 3y^2}{x^2 + y^2} \leq M.$$

3. Verify that $x^2 + 2xy + 2y^2$ is a positive definite form. Find the largest possible number m and the smallest possible number M such that for all $(x, y) \neq (0, 0)$

$$m \leq \frac{x^2 + 2xy + 2y^2}{x^2 + y^2} \leq M.$$

4. Prove that if the discriminant of a quadratic form $Q(x, y)$ is 0 the then the form is semidefinite.

5 Extrema of Functions of Two Variables

The reader will recall that if a is either a maximizing or a minimizing location of the function $f(x)$, then

$$f'(a) = 0.$$

Moreover, a is maximizing or minimizing according as $f''(a)$ is negative or positive. If $f''(a)$ is zero, then no conclusion can be drawn. We now go on to investigate the extension of these concepts to functions of two variables.

Let $f : D \subset \Re^2 \rightarrow \Re$ be a function. Then f is said to have a *relative maximum* at (a, b) provided there is a region $R \subset D$, containing (a, b) in its interior, such that

$$f(a, b) \geq f(x, y) \quad \text{for all} \quad (x, y) \in D.$$

It is clear that each of the surfaces of Figures 6 and 7 has a relative maximum at the origin.

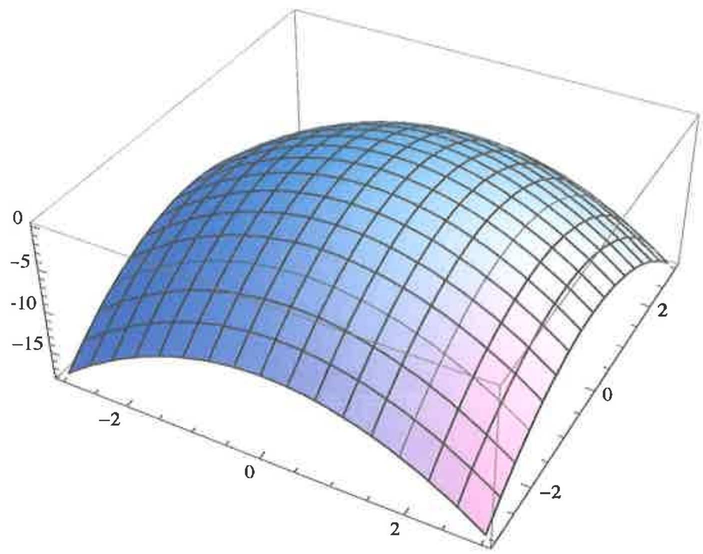

Figure 6: An absolute maximum

Let $f : D \subset \Re^2 \to \Re$ be a function. Then f is said to have an *absolute maximum* at (a, b) provided

$$f(a,b) \geq f(x,y) \quad \text{for all} \quad (x,y) \in D.$$

The relative maximum of Figure 6 at the origin is also an absolute maximum. However, the relative maximum of Figure 7 is not absolute since the points of the surface that are far from the origin are higher than the point at the origin.

Let $f : D \subset \Re^2 \to \Re$ be a function. Then f is said to have a *relative minimum* at (a, b) provided there is a region $R \subset D$, containing (a, b) in its interior, such that

$$f(a,b) \leq f(x,y) \quad \text{for all} \quad (x,y) \in D.$$

It is clear that each of the surfaces of Figures 8 and 9 has a relative minimum at the origin.

Let $f : D \subset \Re^2 \to \Re$ be a function. Then f is said to have an *absolute minimum* at (a, b) provided

$$f(a,b) \leq f(x,y) \quad \text{for all} \quad (x,y) \in D.$$

The relative minimum of Figure 8 at the origin is also an absolute minimum. However, the relative minimum of Figure 9 is not absolute since the points of the surface that are far from the origin are lower than the point at the origin. The *extrema* are its maxima and minima.

Let the function f be differentiable at (a, b) and suppose it has a relative maximum there. Then, for all sufficiently small h

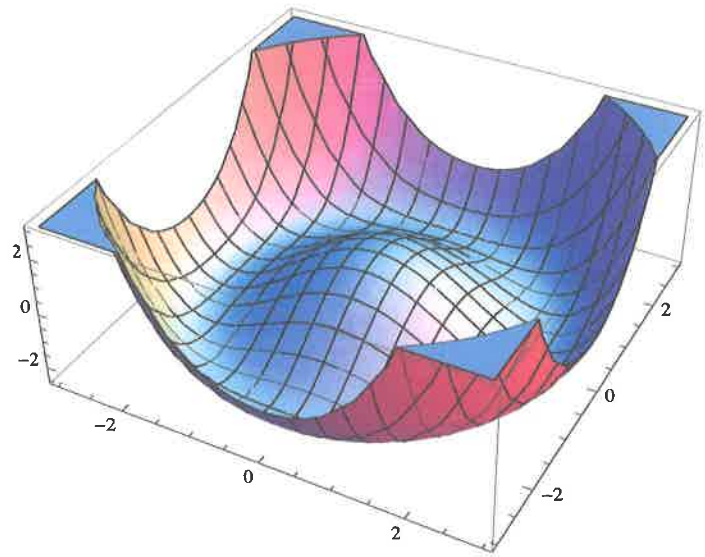

Figure 7: A relative maximum

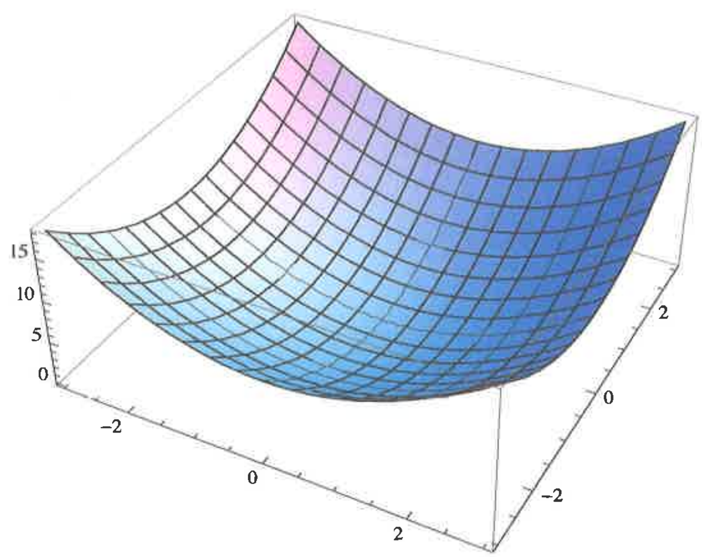

Figure 8: An absolute minimum

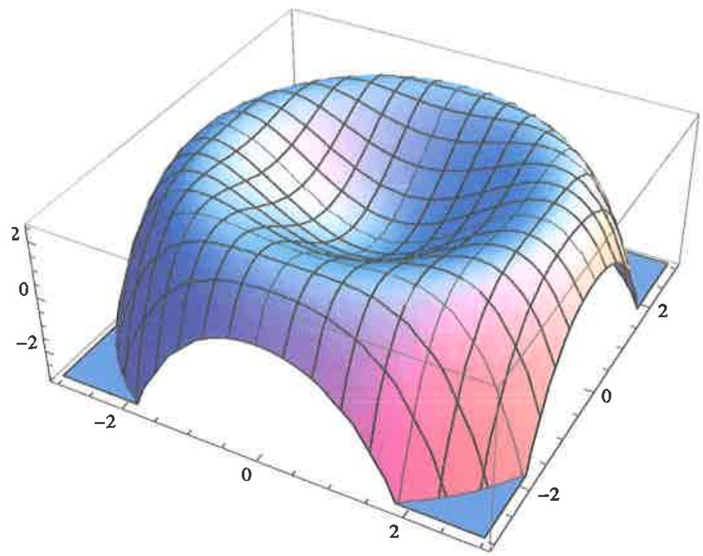

Figure 9: A relative minimum

$$f(a+h,b) \geq f(a,b).$$

and hence

$$\frac{f(a+h,b)-f(a,b)}{h} \begin{cases} \geq 0 & \text{if } h>0 \\ \leq 0 & \text{if } h<0 \end{cases}$$

Consequently, if

$$\lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

exists it must be both non-negative and non-positive, or, in other words, it must be zero. Since f was assumed to be differentiable, it follows that

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} = 0.$$

Similarly,

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h} = 0$$

Similar arguments lead to the same conclusion regarding the the partial derivatives of f at a minimizing location. In summary:

Proposition 5.1 *If (a,b) is an extremizing point of the function f, then*

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

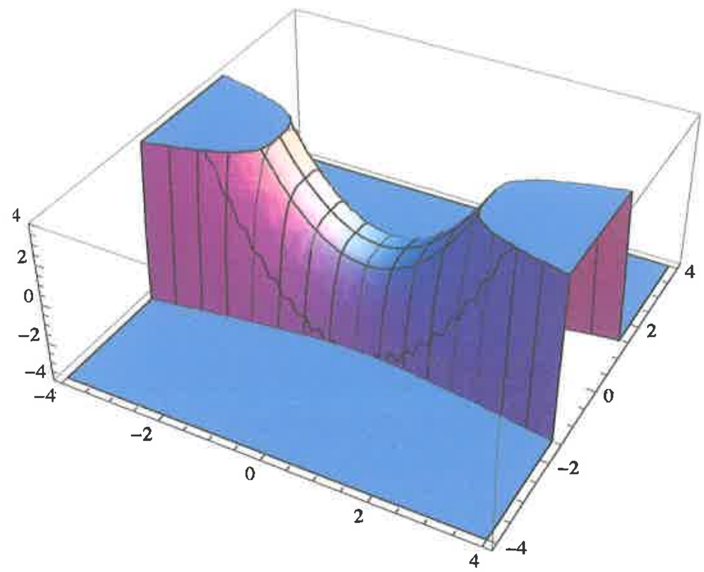

Figure 10: A non-extremal point

□

The converse of this proposition is false, as is demonstrated by Figure 10 which consists of the graph of $f(x,y) = x^2 - y^2$. It is clear that the origin is not an extremal point and yet both partial derivatives there are 0. We shall now describe a 2-dimensional analog of the second derivative test for maxima and minima.

Theorem 5.2 *Let* $z = f(x,y) : D \subset \Re^2 \to \Re$ *such that*

$$f_x(a,b) = f_y(a,b) = 0.$$

1. If $f_{xx}(a,b) > 0$ *and* $f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b) < 0$ *then* f *has a relative minimum at* (a,b).

2. If $f_{xx}(a,b) < 0$ *and* $f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b) < 0$ *then* f *has a relative maximum at* (a,b).

3. If $f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b) > 0$ *the* f *has neither a relative minimum nor a relative maximum at* $(a,\ b)$.

PROOF: Given any constant c the extremal points of $f(x,y)$ as well as its derivatives are the same as those of $f(x,y) - c$. Hence we may assume that

$$0 = f(a,b) = f_x(a,b) = f_y(a,b).$$

Similarly, by the chain rule, coordinate translations have no effect on the derivatives of f and hence we may assume that

$$(a,b) = (0,0).$$

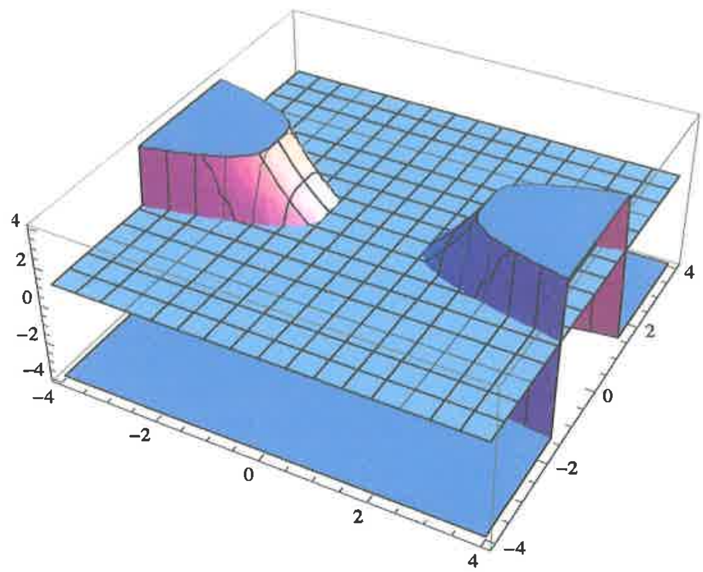

Figure 11: A horizontal cross-section above a point

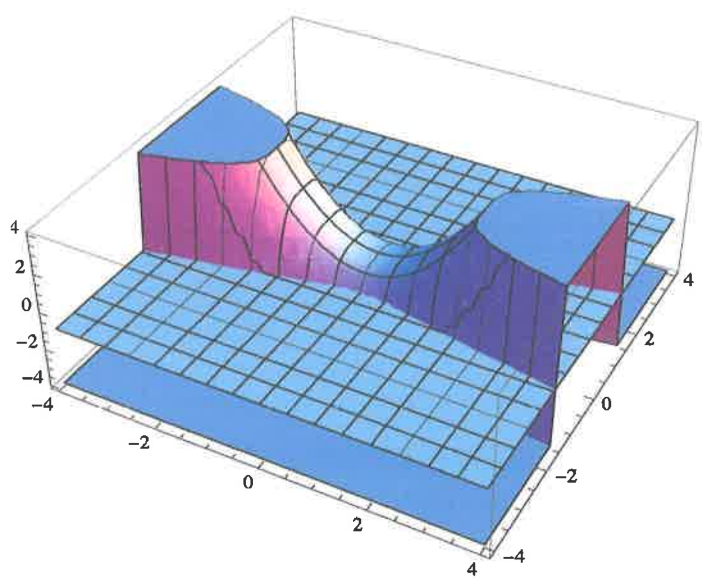

Figure 12: A horizontal cross-section below a saddle point

By Taylor's Theorem 4.3.5, or just Eq'n (13) of Chapter 7,

$$f(x,y) = Ax^2 + Bxy + Cy^2 + R(x,y)$$

where

$$A = \frac{f_{xx}(0,0)}{2}, \quad B = f_{xy}(0,0), \quad C = \frac{f_{yy}(0,0)}{2},$$

$$|R(x,y)| < K(|x|+|y|)^3 \text{ for some constant } K.$$

Set

$$Q(x,y) = Ax^2 + Bxy + Cy^2$$

so that

$$f(x,y) = Q(x,y) + R(x,y). \tag{18}$$

We turn to part 1 of the theorem. The stated inequalities are tantamount to saying that the form Q above is positive definite. By Lemma 4.3 there exists a positive real number m such that for all $(x,y) \neq (0,0)$,

$$Q(x,y) \geq m(x^2+y^2).$$

If we restrict to the subdomain

$$0 \leq |x|, |y| < \frac{m}{8K}$$

then, with the help of Exercise 1,

$$\begin{aligned} |R(x,y)| &< K(|x|+|y|)^3 = K(|x|+|y|)^2(|x|+|y|) \\ &\leq 2K(x^2+y^2)(|x|+|y|) \leq 2K(x^2+y^2) \cdot 2\frac{m}{8K} \\ &= \frac{m}{2}(x^2+y^2) \end{aligned}$$

Hence by, Equation 18,

$$f(x,y) \geq m(x^2+y^2) - \frac{m}{2}(x^2+y^2) > 0 = f(0,0)$$

and it follows that $f(x,y)$ does indeed have a relative minimum at the origin. This concludes the proof of part 1.

Part 2 is relegated to Exercise 2.

For the proof of part 3 we note by Lemma 4.2b that there exist points (x_1,y_1) and (x_2,y_2) such that

$$Q(x_1,y_1) < 0 \quad \text{and} \quad Q(x_2,y_2) > 0.$$

Set

$$m = \frac{1}{2}\min\left\{-\frac{Q(x_1,y_1)}{x_1^2+y_1^2}, \frac{Q(x_2,y_2)}{x_2^2+y_2^2}\right\}$$

so that

$$Q(x_1, y_1) < -m(x_1^2 + y_1^2) \quad \text{and} \quad Q(x_2, y_2) > m(x_2^2 + y_2^2). \tag{19}$$

Again it is necessary to restrict attention to the subdomain

$$0 \le |x|, |y| < \frac{m}{8K} \tag{20}$$

If either (x_1, y_1) or (x_2, y_2) is outside this subdomain, it can, by Lemma 4.4, be replaced by a point that does satisfy both Eq'ns (19) and (20).

Then

$$\begin{aligned} f(x_1, y_1) &= Q(x_1, y_1) + R(x_1, y_1) \\ &< -m(x_1^2 + y_1^2) + 2K(x_1^2 + y_1^2)(|x_1| + |y_1|) \\ &< -m(x_1^2 + y_1^2) + 2K(x_1^2 + y_1^2) \cdot 2\frac{m}{8K} \\ &= -m(x_1^2 + y_1^2) + \frac{m}{2}(x_1^2 + y_1)^2 < 0. \end{aligned}$$

and

$$\begin{aligned} f(x_2, y_2) &= Q(x_2, y_2) + R(x_2, y_2) \\ &> m(x_2^2 + y_2^2) - 2K(x_2^2 + y_2^2)(|x_2| + |y_2|) \\ &> m(x_2^2 + y_2^2) - 2K(x_2^2 + y_2^2) \cdot 2\frac{m}{8K} \\ &= m(x_2^2 + y_2^2) - \frac{m}{2}(x_2^2 + y_2^2) > 0. \end{aligned}$$

Hence, there exist points arbitrarily close to (0,0) where $f(x, y)$ is positive and points arbitrarily close to (0,0) where $f(x, y)$ is negative. Consequently $f(x, y)$ assumes neither a minimum nor a maximum at this point.

Q.E.D.

The proof of the 2-dimensional max/min theorem above is somewhat technical but it can be given a geometrical explanation. Suppose the function f has a (relative) maximum, say 0, at (a, b). Then the cross-section of any horizontal plane with the graph of f near the point $(a, b, 0)$ consists either of a small closed loop, a single point, or an empty set, according as the plane lies below that tangent plane at this point, passes through it, or lies above it. Thus, this cross section is ellipse-like. A similar observation holds for minimum points. On the other hand, if the partial derivatives at (a, b) are zero and it is neither a relative maximum nor a minimum, then the tangent plane at (a, b) is still horizontal and planes that are parallel to it will intersect the surface in hyperbola-like curves (see Figures 11 and 12).

EXERCISES 8.5

1. Prove that if x and y are any real numbers then

$$(|x| + |y|)^2 \leq 2(x^2 + y^2).$$

2. Find the extrema of the following functions:

a. $f(x,y) = x^2+4y^2-4x+2y$ b. $f(x,y) = x^2+xy-y^2$ c. $f(x,y) = x^2-xy+y^2$

d. $f(x,y) = x^2 + 2xy + y^2$ e. $f(x,y) = (x^2 - 2x)(y^2 - 2y)$

f. $f(x,y) = (x^3 - 3x)(y^2 - 2y)$ g. $f(x,y) = (x^3 - 3x)(y^3 - 3y)$

h. $f(x,y) = x^2 - xy + y^3$ i. $f(x,y) = x^2 - xy + y^4$

j. $f(x,y) = 2x - y + x^2 - xy + y^3$ k. $f(x,y) = (2x^3 - 3x^2)(2y^3 - 3y^2)$

6 Constrained Extrema

Many natural optimization problems have built-in constraints on the variables. If the question is of a geometrical nature, the variables may have to be nonnegative; if the problem regards a production problem, the variables may be limited by either the resources or the capacity of the manufacturing facility.

Example 6.1 *For which point* (x, y) *on the ellipse*

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

is the value of the function $f(x,y) = x + 2y$ *as large as possible? For which point is it as small as possible?*

The level curves $x + 2y = c$ are sketched in Figure 13 for some values of c as is the ellipse. These level curves have in general the property that any transversal curve, in particular also the given ellipse, encounters successive level curves of consistently increasing or consistently decreasing c's. The exception to this rule are those level curves that are tangent to the ellipse. These are the points A and B. Another example of this phenomenon is illustrated in Figure 14 which shows how the level curves of the function $x^2 + xy + 4y^2$ are intersected by the unit circle. The critical points are A, B, C, D.

It is clear that the defining property of these critical points is that at these points the level curve is tangent to the constraint curve. This means that these two curves share the same tangent line. Hence they also share the same normal and this is the characteristic that can be used to locate the optimal points.

Let $f(x,y)$ be an arbitrary differentiable function which attains a maximum or minimum at (a, b) subject to the constraint

$$g(x,y) = c.$$

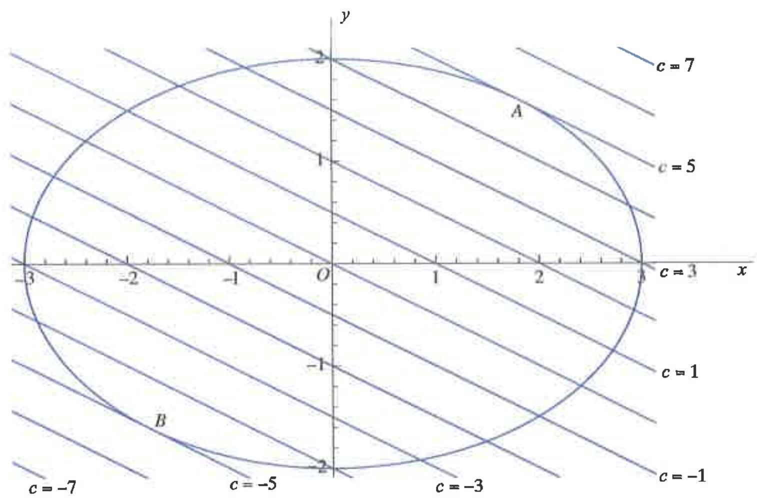

Figure 13: Locating the constrained extrema

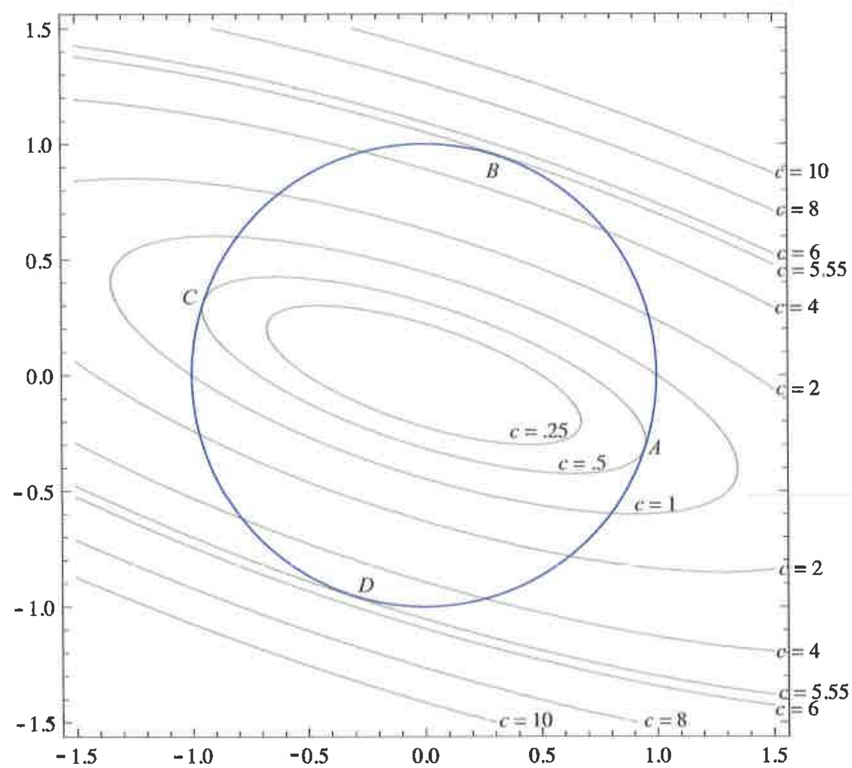

Figure 14: Locating the constrained extrema

We know from Section 2.6 that the gradient

$$\nabla f(a, b)$$

is orthogonal to the level curve of $f(x, y)$ at the point (a, b) which has the equation

$$f(x, y) = f(a, b).$$

Similarly, the gradient $\nabla g(a, b)$ is orthogonal to the level curve

$$g(x, y) = g(a, b).$$

Since it was argued that the two level curves are tangent to each other, it follows that these two gradients have the same directions. In other words, there exists a real number λ, called the *Lagrange multiplier*, such that

$$\lambda \nabla f(a, b) = \nabla g(a, b). \tag{21}$$

Returning to Example 6.1 note that constraining the point to lie on the given ellipse is tantamount to the constraint

$$\frac{a^2}{9} + \frac{b^2}{4} = 1 \tag{22}$$

so that

$$g(x, y) = \frac{x^2}{9} + \frac{y^2}{4}.$$

Eq'n(21) becomes

$$\lambda(1, 2) = \left(\frac{2a}{9}, \frac{2b}{4}\right)$$

or

$$a = \frac{9}{2}\lambda, \qquad b = 4\lambda. \tag{23}$$

Thus, Eq'ns (22, 23) constitute a system of three equations in the three unknowns a, b, and λ. This is a non-trivial task. In general, the following strategy seems to work well for the systems of equations that arise in constrained optimization problems:

Express each of the other unknowns in terms of λ and substitute these expressions into the given constraint $g(x, y) = c$.

Accordingly, we substitute

$$a = \frac{9}{2}\lambda \quad \text{and} \quad b = 4\lambda$$

into Eq'n (22) to obtain

$$\frac{9}{4}\lambda^2 + 4\lambda^2 = 1$$

which implies that

$$\lambda = \pm\frac{2}{5}.$$

It follows from Eq'n (23) that the object function $f(x, y) = x + 2y$ assumes its extremal values at

$$\left(\frac{9}{5}, \frac{8}{5}\right) \quad \text{and} \quad \left(-\frac{9}{5}, -\frac{8}{5}\right).$$

Since

$$f\left(\frac{9}{5}, \frac{8}{5}\right) = 5 > -5 = f\left(-\frac{9}{5}, -\frac{8}{5}\right)$$

it follows that $x + 2y$ assumes its maximum at the first of these points and its minimum at the second.

Actually, Eq'n (21) can be replaced by the logically equivalent

$$\nabla f = \lambda \nabla g. \tag{24}$$

The judicious choice may simplify the resulting equations somewhat.

Example 6.2 *Find the points on the ellipse*

$$2x^2 + 4xy + 3y^2 = 1 \tag{25}$$

where $x^2 - y^2$ is, respectively, minimum and maximum.

The problem calls for the points that extremize

$$f(x, y) = x^2 - y^2$$

subject to the constraint

$$g(x, y) = 2x^2 + 4xy + 3y^2 = 1.$$

Here Eq'n (24) becomes

$$(4a + 4b, 4a + 6b) = \lambda(2a, -2b)$$

which yields the system

$$\begin{cases} (2 - \lambda)a + 2b = 0 \\ 2a + (3 + \lambda)b = 0 \end{cases} \tag{26}$$

This system has a nontrivial solution if and only if

$$\det\begin{pmatrix} 2 - \lambda & 2 \\ 2 & 3 + \lambda \end{pmatrix} = 0$$

which means that

$$(2 - \lambda)(3 + \lambda) - 2 \cdot 2 = 0 \quad \text{or} \quad \lambda^2 + \lambda - 2 = 0.$$

This equation has the solutions $\lambda = 1, -2$.

If $\lambda = 1$ then the system (26) yields $a = -2b$ and substitution into (25) gives

$$b = \pm\frac{1}{\sqrt{3}}, \quad a = \mp\frac{2}{\sqrt{3}} \quad \text{or} \quad (a,b) = \pm\left(\frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

If $\lambda = -2$ then the system (26) yields $b = -2a$ and substitution into (24) gives

$$a = \pm\frac{1}{\sqrt{6}}, \quad b = \mp\frac{2}{\sqrt{3}} \quad \text{or} \quad (a,b) = \pm\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right).$$

Substitution of these points into $f(x,y)$ demonstrates that the first two points maximize and the second pair minimize values of f.

The same method resolves constrained optimization with more independent variables. If there are, say, three variables, then the level curves are replaced by level surfaces. Optimality still entails tangency (of surfaces) and Eq'ns (21) and (24) hold again.

Example 6.3 *Find the dimensions of the largest (in terms of volume) box that can be inscribed in the surface*

$$4x^2 + 3y^2 + 2z^2 = 288.$$

Let (x,y,z) be the vertex of the box in the first octant. We seek to maximize

$$f(x,y,z) = xyz$$

subject to the constraint

$$g(x,y) = 4x^2 + 3y^2 + 2z^2 - 288 = 0$$

Eq'n (21) implies that

$$(bc, ca, ab) = \lambda(8a, 6b, 4c)$$

$$a = \frac{bc}{8\lambda}, \quad b = \frac{ca}{6\lambda}, \quad c = \frac{ab}{4\lambda}$$

The multiplication of each pair of these equations yields, after some terms are cancelled,

$$a^2 = 24\lambda^2, \quad b^2 = 32\lambda^2, \quad c^2 = 48\lambda^2.$$

When these are substituted into the constraint we obtain

$$4 \cdot 24\lambda^2 + 3 \cdot 32\lambda^2 + 2 \cdot 48\lambda^2 = 288$$

or

$$\lambda = \pm 1$$

Since a, b, c must be positive, we have

$$a = 2\sqrt{6}, \quad b = 4\sqrt{2}, \quad c = 4\sqrt{3}.$$

Thus, the dimensions that maximize the volume are

$$4\sqrt{6}, 8\sqrt{2}, 8\sqrt{3}.$$

Suppose the variables are bound by two constraints, say

$$g(x, y, z) = c \quad \text{and} \quad h(x, y, z) = d.$$

Their level surfaces intersect in a curve, say C. If $\vec{t}$ is the tangent vector of C, it must be orthogonal to both ∇g and ∇h. On the other hand, the tangency of C to the level surfaces of f at the critical points implies that $\vec{t}$ is also orthogonal to ∇f. In general, ∇g and ∇h of the plane orthogonal to $\vec{t}$ and so there exist numbers λ and μ such that

$$\nabla f = \lambda \nabla g + \mu \nabla h. \tag{27}$$

This translates to three equations in the five unknowns a, b, c, λ, μ. Two more equations are provided by the constraints.

Example 6.4 *Find the extremal values of* $4x + 2z$ *subject to the constraints* $x^2 + y^2 = 1$ *and* $y^2 + z^2 = 2$.

Equation (27) becomes

$$(4, 0, 2) = \lambda(2x, 2y, 0) + \mu(0, 2y, 2z)$$

or

$$4 = 2\lambda x, \quad 0 = (2\lambda + 2\mu)y, \quad 2 = 2\mu z.$$

If $y = 0$ then $x = \pm 1$ and $z = \pm\sqrt{2}$, which yields the points

$$(\pm 1, 0, \pm\sqrt{2}).$$

If $y \neq 0$ then

$$x = \frac{2}{\lambda}, \quad \lambda + \mu = 0 \quad z = \frac{1}{\mu}.$$

The elimination of y from the constraints yields

$$z^2 - x^2 = 1$$

or

$$\frac{1}{\mu^2} - \frac{4}{\lambda^2} = 1$$

Since $\mu = -\lambda$ this equation becomes

$$1 = \frac{1}{\lambda^2} - \frac{4}{\lambda^2} < 0$$

which is clearly impossible. Thus, the critical points are amongst the points (28) above. It is clear that $(1, 0, \sqrt{2})$ maximizes $4x+2z$ whereas $(-1, 0, -\sqrt{2})$ minimizes it.

EXERCISES 8.6

In Exercises 1-12 find the location and extremal values of the function $f(x, y)$ subject to the constraint $g(x, y) = c$.

1. $f(x,y) = x + y, g(x,y) = 2x^2 + 5xy + 3y^2 - 1$
2. $f(x,y) = 2x - 3y, g(x,y) = 2x^2 + 3xy + y^2 - 2$
3. $f(x,y) = xy^2, g(x,y) = x^2 + y^2 - 9$
4. $f(x,y) = x^2 + 2y^2, g(x,y) = 2x^2 + 2xy + 3y^2 - 1$
5. $f(x,y) = x^2 + 2y^2, g(x,y) = 3x^2 + 4xy + 4y^2 - 2$
6. $f(x,y) = xy, g(x,y) = x^2 + 5xy + 4y^2 - 3$
7. $f(x,y,z) = x + y + z, g(x,y) = x^2 + 2y^2 + 3z^4 - 5$
8. $f(x,y,z) = x + 2y + 3z, g(x,y) = x^2 + y^2 + z^2 - 10$
9. $f(x,y,z) = x^2 + y^2 + z^2, g(x,y,z) = x^2 + 2y^2 + 3z^2$
10. $f(x,y,z) = yz + zx + xy, g(x,y,z) = x^2 + 2y^2 + 3z^2$
11. $f(x,y,z) = xyz, g(x,y,z) = x^2 + 2y^2 + 3z^2$
12. $f(x,y,z) = x^2yz, g(x,y,z) = x^2 + 2y^2 + 3z^2$
13. An open top box is to be made out of 12 meter^2 of material. Find the dimensions that will maximize its volume.
14. An open top box is to have a capacity of V cubic ft. Find the dimensions that will minimize the construction materials.
15. Use the methods of this section to prove that of all the rectangles of fixed perimeter p the square one has the maximum area.
16. Use the methods of this section to prove that of all the rectangles of fixed area a the square one has the minimum perimeter.
17. Use the methods of this chapter to prove that of all the boxes inscribed in a sphere, the cube has the maximum volume.
18. Use the methods of this chapter to prove that of all the boxes inscribed in a sphere, the cube has the maximum surface area.
19. Suppose

$$f(x,y) = Lx^2 + Mxy + Ny^2, \qquad g(x,y) = Ex^2 + Fxy + Gy^2 - c$$

Prove that every Lagrange multiplier λ that arises from extremizing $f(x, y)$ subject to the constraint $g(x, y) = c$ satisfies the quadratic equation

$$(EG - F^2)\lambda^2 - (EN + GL - 2FM)\lambda + (LN - M^2) = 0.$$

20. Assume that the charge for the shipment of a box is proportional to $f(x, y) = xy$ where x denotes the length of the box and y denotes the circumference of the face perpendicular to the direction of this length. Find the dimensions that minimize the charge of a box if

a. the box is to have volume V;

b. the surface area of the box is to be S.

21. Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \ldots x_n}$$

subject to the constraint

$$x_1 + x_2 + \ldots + x_n = c$$

where $x_1, x_2, \ldots, x_n, c$ are non negative numbers. Deduce that

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

22. Find the extremal values of $x + 2z$ subject to the constraints $x^2 + y^2 = 1$ and $y^2 + z^2 = 2$.

23. Find the extremal values of $Ax+By+Cz$ subject to the constraints $x^2 + y^2 = r^2$ and $y^2 + z^2 = R^2$, where A, B, C, r, R are arbitrary.